Second Midterm Review Solutions

Problem 1: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -2 & 0 \end{bmatrix}.$$

and the vector

$$b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(1) Find $p = Ax$ that minimizes $\|Ax - b\|$.

Solution: Note that the solution p minimizing the distance, is the orthogonal projection of b onto $V = C(A)$ the column space of A.
The columns of A are linearly independent, so we can use this to compute the projection matrix $P_V = A(A^TA)^{-1}A^T$. Hence we can compute p as follows

$$p = A \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = A\frac{1}{9} \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -2 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(2) Find x that minimizes $\|Ax - b\|$.

Solution: Note that $p = A((A^TA)^{-1}A^Tb)$, so we can use $x = (A^TA)^{-1}A^Tb$. This is unique as the columns of A are linearly independent. Thus from the above computation

$$x = (A^TA)^{-1}A^Tb = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Problem 2: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ -1 & -3 \\ -1 & -3 \end{bmatrix}$$

(1) Use Gram-Schmidt to find the factorization $A = QR$.

Solution: Denote by v_1 and v_2 the 2 columns of A. First we rescale v_1 to get:

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Now we get an orthogonal vector to q_1 by

$$q_2' = v_2 - (q_1 \cdot v_2)q_1 = v_2 - 5q_1 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Note this vector is already normalized, so $q_2 = q_2'$. These operations can be writen as

$$Q = A \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} E_{12}^{(-5)}$$

So we can rewrite

$$A = QE_{12}^{(5)} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$$

(2) Check that the matrix in (1) satisfies $Q^TQ = I$

Solution:

$$Q^TQ = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & - & -1 & -1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix} = I$$

Problem 3: Consider the linear transformation

$$\phi : \mathbb{R}^3 \to \mathbb{R}^2$$

such that:

- $\phi(e_1) = 3e_1 + 1e_2$
- $\phi(e_2) = 2e_1$
- $\phi(e_3) = e_1 + e_2$

Here recall that we denote by e_i the standard basis.

(1) Find the matrix A of ϕ with respect to the standard basis.

Solution: The above equations give us exactly the first second and third columns of the matrix A respectively, so we get

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(2) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and let $w_1 = \phi(v_1)$ and $w_2 = \phi(v_2)$. What is the matrix B of ϕ with respect to the bases $\{v_i\}$ and $\{w_j\}$.

Solution:

$$w_1 = \phi(v_1) = Av_1 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$w_2 = \phi(v_2) = Av_2 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Further we compute

$$\phi(v_3) = Av_3 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

Consider the change of basis formula $B = W^{-1}AV$ for

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$W = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

So we compute

$$W^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & -1 \end{bmatrix}$$

So we get

$$B = W^{-1}AV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that this makes sense as $\phi(v_1) = w_1$, $\phi(v_2) = w_2$ and $\phi(v_3) = 0$.

Problem 4: Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(1) Find the determinant using the cofactor formula along the first row.

Solution: We compute the required cofactors first

$$C_{11} = (-1)^2 det(M_{11}) = det([d]) = d$$

$$C_{12} = (-1)^3 det(M_{12}) = det([c]) = -c$$

So the cofactor formula becomes

$$det(A) = aC_{11} + bC_{12} = ad - bc$$

(2) Find the determinant using the cofactor formula along the second row.

Solution: We compute the required cofactors first

$$C_{21} = (-1)^3 det(M_{11}) = det([b]) = -b$$

$$C_{22} = (-1)^4 det(M_{22}) = det([a]) = a$$

So the cofactor formula becomes

$$det(A) = cC_{21} + aC_{22} = -bc + ad$$

(3) Use Cramer's rule to find the inverse of the above matrix.

Solution: Note that from the above computations we have computed all cofactors, so we can put them in the cofactor matrix

$$X = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus we get the inverse is given by

$$A^{-1} = \frac{1}{det(A)} X = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Problem 5: Consider the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

(1) Use the cofactor formula to compute the determinant.

Solution: We expand along th first row as this has a zero. We hence have

$$det(A) = 0 * C_{11} + 1 * C_{12} + 2 * C_{13} = -(7 - 3) + 2(5 - 6) = -6$$

(2) Use row operations to compute the determinant.

Solution: We first do Gaussian elimination. Note that to start we need to swap the first 2 rows, so

$$A \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 5 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Note that here we have only swapped rows once, so we get

$$det(A) = -det(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}) = -6$$

Here we use the determinant of upper triangular matrices is the product of diagonal entries

(3) Use the large 3! formula to compute the determinant.

Solution: We write all the terms of the formula to get

$$det(A) = 0*2*7 + 1*1*3 + 2*1*5 - 0*1*5 - 1*1*7 - 2*2*3 = 0 + 3 + 10 - 0 - 7 - 12 = -6$$

Problem 6 Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & -1 & 3 \end{bmatrix}$$

(1) Compute the eigenvalues of the matrix.

Solution: Recall that to consider the eigenvalues we need to consider the zeroes of $det(A - \lambda I)$, so we compute

$$det(A - \lambda I) = det(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & -1-\lambda & 0 \\ -1 & -1 & 3-\lambda \end{bmatrix}) = (1-\lambda)(-1-\lambda)(3-\lambda)$$

Here again we use the determinant of a lower triangular matrix is the product of the diagonal entries.
Thus from the above we see that the zeroes of the above are given by $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 3$

(2) Compute eigenvectors for the above eigenvalues. Is there an eigenvector that is particularly easy?

Solution: First we compute the eigenvector of $\lambda_1 = 1$, ie we need to find elements in the nullspace of

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Note that the sum of the rows are 0, so we get an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now we compute the eigenvector of $\lambda_2 = -1$, so we need to find elements in the nullspace of

$$A + I = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & 4 \end{bmatrix}$$

Here we can get an eigenvector $\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$.

Finally we compute the eigenvector of $\lambda_3 = 3$, so we need to find elements in the nullspace of

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -4 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

Note that the last colomn is 0, so we get an eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that this eigenvector always works for a lower triangular matrix.