First Midterm Review Problems

Problem 1: Consider the matrix

$$A = \begin{bmatrix} 3 & 6 & 3 & 9 \\ -2 & -4 & 0 & 2 \end{bmatrix}.$$

(1) Compute the reduced row echelon form of A.

Solution: The reduced row echelon form may be calculated via the sequence of row operations

$$\begin{bmatrix} 3 & 6 & 3 & 9 \\ 2 & -4 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -4 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & 3 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

(2) What is a basis for the column space of A? What is a basis for the nullspace of A? What is the rank of A?

Solution: The column space of A has basis

$$\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}.$$

The rank of A is thus 2. The null space consists of the set of vectors

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

such that both

$$a + 2b - d = 0, \text{ and}$$

$$c + 4d = 0.$$

The 2nd and 4th columns of the row reduction of A do not contain pivots, so b and d are free variables. Setting $b = 1$ and $d = 0$ yields $a = -2$ and $c = 0$. Setting $b = 0$ and $d = 1$ yields $a = 1$ and $c = -4$. A basis for the null space of A is thus

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

(3) Write down a general solution to the following system of 2 equations in 4 variables:

$$3a + 6b + 3c + 9d = 0$$

$$-2a - 4b + 2d = -6.$$

Solution: A general solution looks like

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 9 \\ -3 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix},$$

where α and β are arbitrary numbers. In other words, every solution is given by a particular solution plus an element of the null space of A. While the above expression uses $\begin{bmatrix} 0 \\ 0 \\ 9 \\ -3 \end{bmatrix}$ as the particular solution, many other particular solutions are possible.

(4) Write down a basis for the column space of A^T, and for the nullspace of A^T.

Solution: Since the rank of A is 2, the rank of A^T is also 2. Thus, the two columns of A^T (i.e., the two rows of A) are linearly independent, and a basis for the column space of A^T is given by

$$\left\{ \begin{bmatrix} 3 \\ 6 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Since A^T has rank 2, but A^T only has 2 columns, the nullspace of A^T is zero-dimensional. Thus,

$$N(A^T) = \{\mathbf{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

A basis for $N(A^T)$ is the empty set. Don't worry if that last sentence is confusing; on a homework or exam it is sufficient simply to say that $N(A^T)$ contains only the origin.

Problem 2: Write down the 4×4 elimination matrix $E_{3,1}^{(-2)}$. What is its inverse?

Solution: We have

$$E_{3,1}^{(-2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and

$$\left(E_{3,1}^{(-2)}\right)^{-1} = E_{3,1}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 3: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 8 & 11 \\ -2 & -4 & -8 & -8 \\ 1 & 1 & 2 & 3 \end{bmatrix}.$$

(1) Compute the LU factorization of A. Recall that L is lower-triangular with 1s along the diagonal, and U is upper triangular. The matrix U is obtained by bringing A into row echelon form (but not into reduced row echelon form).

Solution: We use the following sequence of row operations to bring A into row echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 8 & 11 \\ -2 & -4 & -8 & -8 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{E_{21}^{(-2)}} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ -2 & -4 & -8 & -8 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{E_{31}^{(2)}} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & -2 & -4 & -2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{E_{41}^{(-1)}} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{E_{32}^{(1)}} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2) Write L as a product of 4×4 elimination matrices.

Solution: $L = E_{21}^{(2)} E_{31}^{(-2)} E_{41}^{(1)} E_{32}^{(-1)}$

(3) What is a basis for the column space of A? What is the rank of A? Is A invertible?

Solution: A basis for the column space consists of the 1st, 2nd, and 4th columns of A. The rank of A is 3. Since the columns of A are not linearly independent, A is not invertible.

(4) Let A' denote the matrix

$$A' = \begin{bmatrix} -2 & -4 & -8 & -8 \\ 1 & 1 & 2 & 3 \\ 2 & 4 & 8 & 11 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

Construct a $PA' = LU$ decomposition by finding P, L, and U such that P is a permutation matrix, L is lower triangular with 1s on the diagonal, and U is upper triangular.

Solution: We can use

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This makes $PA' = A$, and so we may use the L and U matrices from the first part of this problem.