18.06 FINAL

December 17, 2019 (180 minutes)

Please turn cell phones off completely and put them away.

No books, notes, or electronic devices are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Please write your name on **ALL** pages that you want graded (those will be the ones we scan).

The back sides of the paper will **NOT** be graded (for scratch work only).

Do not unstaple the exam, nor reorder the sheets.

If you need extra space for any problem, use the extra sheets at the back of the book. If you do so, please mention this in the main body of the problem in question, so we know to look.

There are 6 problems, worth 225 points in total.

NAME: ______________________________

MIT ID NUMBER: ______________________________

RECITATION INSTRUCTOR: ______________________________

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

PROBLEM 1 **NAME:** ____________________

In this problem, consider the 4×4 matrix A whose columns are vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4 \in \mathbb{R}^4$:

$$A = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array} \right]$$

which are mutually orthogonal ($\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for all $i \neq j$) and have the following lengths:

$$||\mathbf{a}_1|| = 1, \quad ||\mathbf{a}_2|| = 3, \quad ||\mathbf{a}_3|| = 5, \quad ||\mathbf{a}_4|| = 7$$

(1) Compute $A^T A$. *(5 points)*

(2) Write A as a sum of four rank 1 matrices, and write each of those rank 1 matrices as a column vector times a row vector (the answer will depend on $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$). *(10 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ______________________

(3) Compute the explicit formula for the solution $\mathbf{v} \in \mathbb{R}^4$ to the equation $A\mathbf{v} = \mathbf{b}$. Your answer should only depend on $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b} \in \mathbb{R}^4$. *Hint: remember (1).* *(10 points)*

For the explicit matrix A studied above, consider the matrix:

$$B = A \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(4) Write the third column of B in terms of the columns of A. *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(5) Compute the absolute value of the determinant of the matrix B from above. *(5 points)*

(6) Which two columns of B are orthogonal? How do you know? *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

PROBLEM 2 **NAME:** ____________________

Consider the matrix:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(1) Without doing any computations, what number is $\det C$, and why? *(5 points)*

(2) Compute the eigenvalues λ_1, λ_2, λ_3 of C. Explain your reasoning. *(10 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(3) Use the general method (i.e. Gauss-Jordan elimination to compute nullspaces) in order to compute eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of C corresponding to the eigenvalues λ_1, λ_2, λ_3.

(10 points)

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(4) For the given matrix C and its eigenvalues λ_1, λ_2, λ_3 computed in part (2), calculate:

$$(C - \lambda_1 \cdot I)(C - \lambda_2 \cdot I)(C - \lambda_3 \cdot I)$$

(5 points)

(5) Use part (4) to get formulas for C^3 <u>and</u> C^4 in terms of C^2 and C only (i.e. your formulas should be of the form $C^3 = xC^2 + yC$ for some constants x, y, and similarly for C^4).

(5 points)

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(6) Consider any vector of the form $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$, where α, β, γ are real numbers, and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the eigenvectors of C computed in part (3). Compute:

$$C^n\mathbf{v}$$

in terms of α, β, γ, n and vectors with explicit numbers as entries.

Then deduce an explicit formula for:

$$\lim_{n\to\infty} C^n\mathbf{v}$$

in terms of α, β, γ and vectors with explicit numbers as entries. *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

PROBLEM 3 **NAME:** ____________________

Let $M = \begin{bmatrix} 1 & -1 & 3 & z \\ 0 & -1 & y & -1 \\ x & 2 & -1 & 5 \end{bmatrix}$ and consider the equation (with $\mathbf{v} \in \mathbb{R}^4$ and $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \in \mathbb{R}^3$):

$$M\mathbf{v} = \mathbf{n}$$

We know that the equation has solutions $\mathbf{v} \in \mathbb{R}^4$ if and only if $n_1 + n_2 + n_3 = 0$.

(1) Based on the information given in the previous sentence, what is the column space of M and what are the mystery numbers x, y, z? Explain how you know. *(5 points)*

(2) For which $\mathbf{n} \in \mathbb{R}^3$ is the set $\{\mathbf{v} \in \mathbb{R}^4 \text{ such that } M\mathbf{v} = \mathbf{n}\}$ a subspace of $\mathbb{R}^4$, and why? *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(3) With the explicit values you found for x, y, z, find the complete solution to the equation:

$$M\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(solve this part by Gauss-Jordan elimination, identifying free/pivot variables and all that).

(10 points)

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(4) Let $V \subset \mathbb{R}^3$ be the orthogonal complement of the column space you computed in part (1).

- V is called the ____________________ of the matrix M. *(3 points)*
- Compute the projection matrix P_V that describes projection onto V. *(7 points)*

(5) If N is any other matrix such that the column space of N is contained inside the nullspace of M, then what is the product MN equal to? Justify your answer. *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

PROBLEM 4 **NAME:** ____________________

(1) Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$. Compute $A^T A$, its eigenvalues and eigenvectors. *(10 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ______________________

(2) Compute the SVD of the matrix A from part (1), i.e. write it as:

$$A = U\Sigma V^T$$

where $U^TU = I_4$, $V^TV = I_2$, and the only non-zero entries of Σ are on its diagonal. You should deduce Σ and V from your work in part (1). Afterwards, this information will give you two columns of U, but you must compute the other two by Gram-Schmidt. *(10 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(3) Let $S = A^T A$ be the symmetric 2×2 matrix you computed in part (1), and define:

$$e(x, y) = \begin{bmatrix} x & y \end{bmatrix} S \begin{bmatrix} x \\ y \end{bmatrix}$$

for any two numbers x and y.

- The explicit formula for the quantity above is $e(x, y) =$ ________________ *(3 points)*
- We have $e(x, y) > 0$ for all $(x, y) \neq (0, 0)$ because S is ________________ *(3 points)*
- $\{e(x, y) = 1\}$ is the equation of the following geometric shape in the x, y plane:

____

(3 points)

(4) Obtain a linear transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ which transforms the geometric shape $\{e(x, y) = 1\}$ into a circle centered at the origin (full points for a correct formula for ϕ with explanation of how you got it, half points for a good geometric description of ϕ in words).

(6 points)

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

PROBLEM 5 **NAME:** ________________

(1) Compute the determinant of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Show all the steps. *(10 pts)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(2) Compute $\det \begin{bmatrix} 0 & x & 0 & -1 \\ 2 & -5 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix}$ as a function of x. Show your steps. *(10 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ______________

(3) Pick a complex (non-real) number z and write it in both Cartesian ($z = a + bi$) and polar ($z = re^{i\theta}$) form. In doing so, give the formula which connects r, θ to a, b. *(5 points)*

(4) For the z you just chose, compute $z + \bar{z}$ and $z\bar{z}$, where $\bar{z} = a - bi$ is the conjugate of z. Write down a 2×2 matrix J with real entries, whose eigenvalues are z and $\bar{z}$. *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(5) Consider the solution to the system of differential equations:

$$\begin{bmatrix} \dot{f}(t) \\ \dot{g}(t) \end{bmatrix} = J \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

where J is the matrix you chose in part (4). Any solution to this equation is of the form:

$$f(t) = c_1 \cdot \ldots\ldots\ldots\ldots\ldots\ldots\ldots \;+\; c_2 \cdot \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where c_1 and c_2 are some scalars. Fill the blanks with two explicit functions of t, which you should get from what you already know about the matrix J (no further computations are necessary, but you should say explicitly what these functions have to do with J). *(5 points)*

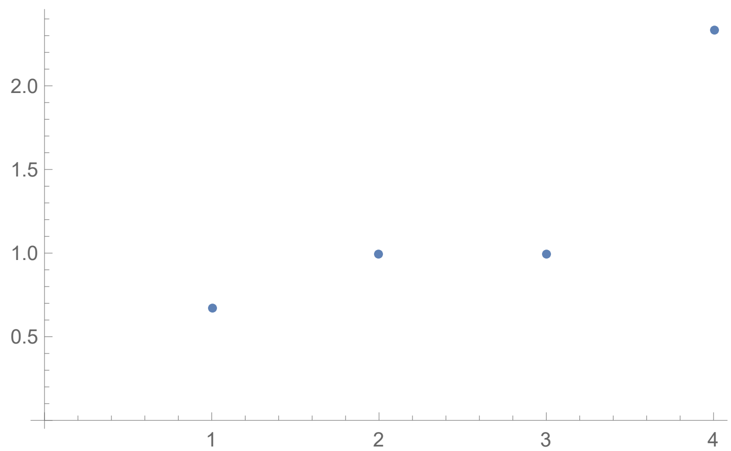
(6) Explain (in words) the qualitative behavior of $f(t)$ from part (5) as $t \to \infty$. *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

PROBLEM 6 **NAME:** ____________________

At times 1, 2, 3, 4, you measure temperatures $\frac{2}{3}$, 1, 1, $\frac{7}{3}$, respectively. These are represented as data points on the following plot (horizontal axis is time, vertical axis is temperature):



The goal is to find numbers a, b such that the line $y = ax + b$ is the best fit for the data points above. Specifically, this means that a and b should be chosen such that the quantity:

$$\Upsilon = \left(a \cdot 1 + b - \frac{2}{3}\right)^2 + (a \cdot 2 + b - 1)^2 + (a \cdot 3 + b - 1)^2 + \left(a \cdot 4 + b - \frac{7}{3}\right)^2 \quad \text{is minimal}$$

(1) Define a 4×2 matrix A, a 2×1 vector $\mathbf{v}$, and a 4×1 vector $\mathbf{b}$ such that:

$$\Upsilon = ||A\mathbf{v} - \mathbf{b}||^2$$

(the entries of A and $\mathbf{b}$ should be numbers, and those of $\mathbf{v}$ should be unknowns). *(5 points)*

(2) Write down the general formula for the solution to the least squares problem: if the columns of the matrix A are independent, then the quantity $||A\mathbf{v} - \mathbf{b}||$ is minimal for:

$$\mathbf{v} = \underline{\qquad\qquad\qquad\qquad}$$

(the answer should be a formula in terms of A and $\mathbf{b}$; don't plug in numbers yet). *(5 points)*

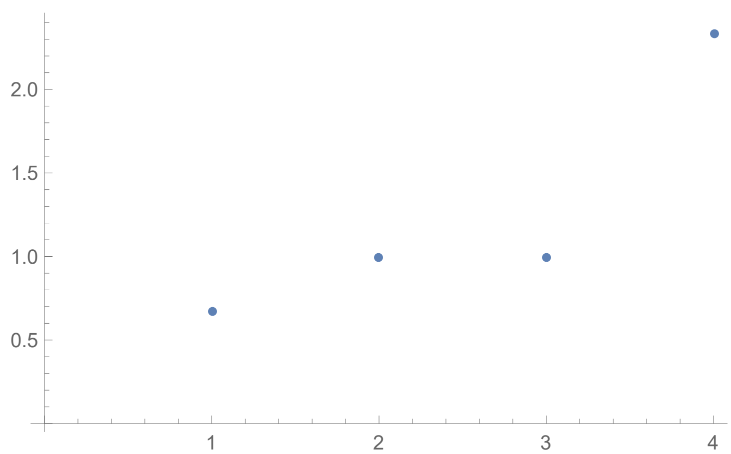
SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

(3) Use the previous two parts to solve for the numbers a and b that give the best line fit for our four data points (i.e. that minimize the quantity Υ). Show your work. *(10 points)*

(4) With the values you just found for a and b, draw the precise line $y = ax + b$ in the graph:



(5 points)

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

NAME: ____________________

Let's look at the aforementioned data points from the point of view of statistics. Put the sample sets for time and temperature in a 4×2 matrix:

$$\mathbf{Z} = \left[\ \mathbf{x} \mid \mathbf{y}\ \right] \qquad \text{where } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \\ \frac{7}{3} \end{bmatrix}$$

The covariance matrix of the sample sets $\mathbf{x} =$"time" and $\mathbf{y} =$"temperature" is given by:

$$K = \begin{bmatrix} K_{\mathbf{xx}} & K_{\mathbf{xy}} \\ K_{\mathbf{yx}} & K_{\mathbf{yy}} \end{bmatrix} = \frac{\mathbf{Z}^T P \mathbf{Z}}{4-1}, \qquad \text{where } P = \frac{1}{4} \cdot \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

A computer tells us that this matrix takes the form (up to two decimals approximation):

$$K = \underbrace{\begin{bmatrix} -0.53 & 1.87 \\ 1 & 1 \end{bmatrix}}_{\text{orthogonal}} \begin{bmatrix} 0.1 & 0 \\ 0 & 2.11 \end{bmatrix} \begin{bmatrix} -0.53 & 1 \\ 1.87 & 1 \end{bmatrix}$$

(5) Using all the information above, for any constants α and β, write the variance of the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ in terms of the matrix K and the vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ *(5 points)*

(6) Using all the information above, find particular constants α and β such that the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ has variance precisely 0.1 (and explain how you know) *(5 points)*

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

Continuation of work on problem number _____ **NAME:** ____________________

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

Continuation of work on problem number _____ **NAME:** ____________________

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

Continuation of work on problem number _____ **NAME:** ____________________

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED

Continuation of work on problem number _____ **NAME:** ____________________

SCRATCH WORK

THIS SIDE WILL NOT BE GRADED