Recitation 13. December 10

Focus: Fourier Series, Population Dynamics, and Graphs

Any 2π-periodic function $f(x)$ has a Fourier series expansion

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots,$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

and, for each integer $n > 0$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \text{ and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

1. Consider the 2π-periodic square wave, which on the interval $[-\pi, \pi]$ is described by

$$f(x) = \begin{cases} 0, \text{ if } -\pi \leq x \leq 0 \\ 1, \text{ if } 0 < x \leq \pi \end{cases}$$

Compute the Fourier series expansion of $f(x)$.

Solution:

2. In a certain habitat, the number of rabbits r_k and wolves w_k is recorded each year k. It is observed that the quantities obey the following formulae:

   - $r_k = 4r_{k-1} - 2w_{k-1}$.
   - $w_k = r_{k-1} + w_{k-1}$.

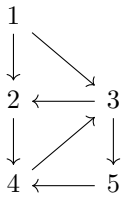
   A) If $r_0 = 4$ and $w_0 = 2$, what are r_{15} and w_{15}?

   B) If $r_0 = 2$ and $w_0 = 2$, what are r_{15} and w_{15}?

   C) What about when $r_0 = 6$ and $w_0 = 4$?

Solution:

3. The adjacency matrix A of the following graph is a 5×5 matrix:



   The entry in row i and column j is 1 if there is an arrow connecting i to j, and it is 0 if $i = j$ or if there is no arrow connecting i to j. Write down the adjacency matrix A, and compute A^2 as well as $(A^2)^2 = A^4$. For each pair (i, j), how many length 4 paths are there from i to j?

Solution: