Recitation 13. December 10

Focus: Fourier Series, Population Dynamics, and Graphs

Any 2π-periodic function $f(x)$ has a Fourier series expansion

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots,$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

and, for each integer $n > 0$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \text{ and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

1. Consider the 2π-periodic square wave, which on the interval $[-\pi, \pi]$ is described by

$$f(x) = \begin{cases} 0, & \text{if } -\pi \le x \le 0 \\ 1, & \text{if } 0 < x \le \pi \end{cases}$$

Compute the Fourier series expansion of $f(x)$.

Solution:
We calculate the various coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} 1dx = 1/2.$$

For each integer $n > 0$, we have:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)dx = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{0}^{\pi} sin(nx)dx = \frac{-cos(n\pi) + \cos(0)}{\pi n}.$$

When n is an integer, $\cos(n\pi) = (-1)^n$, and so $b_n = \frac{1-(-1)^n}{\pi n}$ The Fourier series for the square wave is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n}\sin(nx) = \frac{1}{2} + \frac{2\sin(x)}{\pi} + \frac{2\sin(3x)}{3\pi} + \frac{2\sin(5x)}{5\pi} + \cdots.$$

Just as a remark, we could have predicted that the a_n are 0 in advance, since $f(x) - \frac{1}{2}$ is an odd function like $\sin(nx)$ and unlike $\cos(nx)$.

2. In a certain habitat, the number of rabbits r_k and wolves w_k is recorded each year k. It is observed that the quantities obey the following formulae:

- $r_k = 4r_{k-1} - 2w_{k-1}$.
- $w_k = r_{k-1} + w_{k-1}$.

A) If $r_0 = 4$ and $w_0 = 2$, what are r_{15} and w_{15}?

B) If $r_0 = 2$ and $w_0 = 2$, what are r_{15} and w_{15}?

C) What about when $r_0 = 6$ and $w_0 = 4$?

Solution: This example is discussed on pages 104-105 of the lecture notes. Let M denote the matrix

$$M = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}.$$

Then there is an equation

$$\begin{bmatrix} r_k \\ w_k \end{bmatrix} = M^k \begin{bmatrix} r_0 \\ w_0 \end{bmatrix}.$$

In situation (A), we may note that $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} r_0 \\ w_0 \end{bmatrix}$ is an eigenvector of M of eigenvalue 3. Therefore $r_{15} = 4(3)^{15}$ and $w_{15} = 2(3)^{15}$.

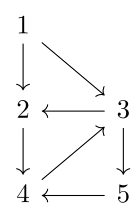
In situation (B), $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} r_0 \\ w_0 \end{bmatrix}$ is an eigenvector of M of eigenvalue 2. Therefore $r_{15} = 2(2)^{15} = 2^{16}$ and $w_{15} = 2(2)^{15} = 2^{16}$.

In situation (C), we may write

$$M^{15} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = M^{15} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + M^{15} \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

and so $r_{15} = 4(3)^{15} + 2^{16}$ and $w_{15} = 2(3)^{15} + 2^{16}$. As this example shows, it can be helpful to write arbitrary vectors in terms of eigenvectors, or alternatively to diagonalize M as in the lecture notes.

3. The adjacency matrix A of the following graph is a 5×5 matrix:



The entry in row i and column j is 1 if there is an arrow connecting i to j, and it is 0 if $i = j$ or if there is no arrow connecting i to j. Write down the adjacency matrix A, and compute A^2 as well as $(A^2)^2 = A^4$. For each pair (i, j), how many length 4 paths are there from i to j?

Solution: We see that

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

The entry of A^4 in row i and column j records the number of length 4 paths from i to j. For example, there are two length 4 paths from 3 to 2, one of which proceeds $3 \to 5 \to 4 \to 3 \to 2$ and the other of which proceeds $3 \to 2 \to 4 \to 3 \to 2$.