Recitation 12. December 3

Focus: Probability, Statistics and Markov Chains
Consider running an experiment n times measuring a certain quantity. We call these values x_1, $x_2 \dots x_n$ ***samples***. The collection of these is known as ***data set***.
The ***mean*** of a data set is given by $\mu = \frac{1}{n}(x_1 + \cdots + x_n)$. The ***variance*** of the data set is $\Sigma = \frac{1}{n-1}((x_1 - \mu)^2 + \cdots + (x_n - \mu)^2)$. Similarly for a second data set $y_1, \dots y_n$ the covariance between x's and y's is given by $\Sigma_{xy} = \frac{1}{n-1}((x_1 - \mu)(y_1 - \nu) + \cdots + (x_n - \mu)(y_n - \nu))$, where ν is the average of the y's.

Let $P = I - \mathbf{o}\mathbf{o}^T/\mathbf{o}^T\mathbf{o}$, where $\mathbf{o} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, ie the projection matrix to the orthogonal complement of $\mathbf{o}$. Let $A = \begin{bmatrix} x_1 & y_1 & z_1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & \dots \end{bmatrix}$ a matrix of different data sets. Then the covariance matrix is computed by $K = \frac{A^T P A}{n-1}$.
A ***Markov matrix*** M is a square matrix with non-negative entries whose columns add up to 1. It models a ***Markov process*** which has n states and after each step the ith column of M gives the probability of moving from the ith state to the other. So if we start with a distribution v computing Mv gives the probability distribution after one step. Using eigenvalues and eigenvectors we can compute the steady state given by the limit $M^k v$ as $k \to \infty$

1. Consider the following measuraments of temperature and pressure (measured in some units) given by $Temp = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $Press = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$
   - Compute the covariance matrix.
   - Find linear combinations of temperature and pressure that are independent

Solution: First we put into a matrix the above samples into a matrix

$$A = \begin{bmatrix} 1 & 6 \\ 2 & 1 \\ -3 & 2 \end{bmatrix}$$

And let $P = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. Then the covariance matrix is given by

$$K = A^T PA/n - 1 = \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}$$

To find independent random variables we need to diagonalize K. Note that the eigenvalues are given by $\lambda_1 = 6$ and $\lambda_2 = 8$ with eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Thus we get that $Temp - Preas$ and $Temp + Press$ are independent random variables

2. Consider three independent random variable X_1, X_2 and X_3. Assume they have variances Σ_1, Σ_2 and Σ_3 and means μ_1, μ_2 and μ_3.
   - What is the covariance matrix of these variables?
   - What is the variance of $X_1 + X_2 + X_3$?
   - What is the covariance of X_1 and $X_1 + X_2 + X_3$?

Solution: Since the random variables are independent their covariance is 0.
Hence we get the covariance matrix is given by

$$K = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{bmatrix}$$

Note the linear combination of the above random variables is given by the vector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence the variance of this random variable is given by $v^T K v = \Sigma_1 + \Sigma_2 + \Sigma_3$.
Similarly the covariance of the two linear combinations are given by $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} K v = \Sigma_1$.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

For some constants a and b Suppose it is a covariance matrix of two random variables X and Y.

- What can you say about a and b?
- What can you say if on top of the above you know that there is some non-trivial linear combination of X and Y that are constant?

Solution: Note that because it is a covariance matrix it is a symmetric positive semidefinite matrix. Hence we know $a = 2$ and since both eigenvalues have to be ≥ 0 we need the determinant to be positive, so $b - 4 \geq 0$. Note that in that case indeed the sum and product of eigenvalues is non-negative hence the matrix is positive semidefinite.
If a nontrivial linear combination of X and Y are constant it means there is some non-zero vector v, such that $v^T A v = 0$. Thus we must have A is not positive definite, but positive semidefinite, hence has a 0 eigenvalue. So in particular the determinant is 0, thus we need $b = 4$. In that case $2X - Y$ is in fact constant.

4. Consider the Markov process with two states up and down.

- If the state is up the next step is equaly likely to be up or down.
- If the state is down the next step is up.

1. Find the Markov matrix describing this Markov process.
2. If we start in the up state, what is the probability distribution after k steps?
3. What is the steady state?

Solution: The Markov matrix above is given by the matrix

$$M = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$$

The kth step from the above is given by $M^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To find this we need to compute the kth power of a matrix, so we need to compute the eigenvalues and eigenvectors. Note that since this is a Markov matrix it has an eigenvalue 1 and hence the other eigenvalue is $-1/2$. The corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So the starting state in terms of the eigenvectors is $\frac{1}{3}(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and hence the state for the kth step is

$$\frac{1}{3}(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1/2)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix})$$

Here we see that the second term goes to 0 as k goes to ∞. Hence we get the steady state is $\frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.