Recitation 9. November 12

Focus: Differential equations. Complex numbers. Symmetric matrices and eigenvalues

Recall the matrix exponential is defined by $e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots + \frac{A^nt^n}{n!} + \cdots$. Using this we can solve systems of linear differential equations of the form $u'(t) = Au(t)$. The solutions to this system are given by $u(t) = e^{At}u_0$. Here u_0 is a constant vector describing the initial solutions.

The imaginary number i is defined by $i^2 = -1$. Using this we can define the complex numbers as expressions $z = a + bi$. These can be summed in the obvious way and multiplied by $(a + bi)(c + di) = ac - bd + (ad + bc)i$. Any degree n polynomial has exactly n roots when counted with multiplicities, but it might have complex numbers. Even real matrices can have complex eigenvalues.

A symmetric matrix $A = A^T$ always has real eigenvalues and has a basis of orthonormal eigenvectors. It follows that we can always diagonalize the matrix A as $A = Q\Lambda Q^T$, where Q is an orthogonal matrix and Λ the matrix of eigenvalues.

1. Consider the following system of differential equations

$$\begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = \epsilon^2 u_1 \end{cases}$$

- Solve this equation using the exponential of a certain matrix A
- What happens when $\epsilon \to 0$, ie when you take the limit of the solution computed above when $\epsilon \to 0$

Solution:

2. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of this matrix. How many eigenvalues are complex? Are there complex conjugate pairs?

Solution:

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   1. Can you find an easy eigenvalue without computing the characteristic polynomial?
   2. Compute all eigenvectors for the above easy eigenvalue
   3. Can you use this to determine the remaining eigenvector and eigenvalue?

Solution:

4. Let n be an odd number. Let A and $n \times n$ real matrix. Prove that this matrix always has a real eigenvalue.

Solution: