Recitation 9. November 12

Focus: Differential equations. Complex numbers. Symmetric matrices and eigenvalues

Recall the matrix exponential is defined by $e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots + \frac{A^nt^n}{n!} + \cdots$. Using this we can solve systems of linear differential equations of the form $u'(t) = Au(t)$. The solutions to this system are given by $u(t) = e^{At}u_0$. Here u_0 is a constant vector describing the initial solutions.

The imaginary number i is defined by $i^2 = -1$. Using this we can define the complex numbers as expressions $z = a + bi$. These can be summed in the obvious way and multiplied by $(a+bi)(c+di) = ac - bd + (ad+bc)i$. Any degree n polynomial has exactly n roots when counted with multiplicities, but it might have complex numbers. Even real matrices can have complex eigenvalues.

A symmetric matrix $A = A^T$ always has real eigenvalues and has a basis of orthonormal eigenvectors. It follows that we can always diagonalize the matrix A as $A = Q\Lambda Q^T$, where Q is an orthogonal matrix and Λ the matrix of eigenvalues.

1. Consider the following system of differential equations

$$\begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = \epsilon^2 u_1 \end{cases}$$

- Solve this equation using the exponential of a certain matrix A
- What happens when $\epsilon \to 0$, ie when you take the limit of the solution computed above when $\epsilon \to 0$

Solution: This system of equations are given by $u'(t) = Au(t) = \begin{bmatrix} 0 & 1 \\ \epsilon^2 & 0 \end{bmatrix} u(t)$. We find eigenvalues and eigenvectors of this matrix. Note that the characteristic polynomial is given by $\lambda^2 - \epsilon^2$, so the eigenvalues are $\pm\epsilon$. Computing the nullspace of $A \pm \epsilon$ we get the eigenvectors are $\begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$ for $\lambda = \epsilon$ and $\begin{bmatrix} 1 \\ -\epsilon \end{bmatrix}$ for $\lambda = -\epsilon$. Thus we can diagonalize the matrix as

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ \epsilon & -\epsilon \end{bmatrix} \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix} \frac{1}{2\epsilon} \begin{bmatrix} \epsilon & 1 \\ \epsilon & -1 \end{bmatrix}$$

Thus we compute

$$e^{At} = \begin{bmatrix} 1 & 1 \\ \epsilon & -\epsilon \end{bmatrix} \begin{bmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{bmatrix} \frac{1}{2\epsilon} \begin{bmatrix} \epsilon & 1 \\ \epsilon & -1 \end{bmatrix} = \begin{bmatrix} \frac{e^{\epsilon t}+e^{-\epsilon t}}{2} & \frac{e^{\epsilon t}-e^{-\epsilon t}}{2\epsilon} \\ \epsilon\frac{e^{\epsilon t}-e^{-\epsilon t}}{2} & \frac{e^{\epsilon t}+e^{-\epsilon t}}{2} \end{bmatrix}$$

And thus the solution is $u(t) = e^{At}u_0$ for some constant vector u_0.
Now letting $\epsilon \to 0$ we get

$$e^{At} \to \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Note that this is the exponantial for the matrix $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ which is the limit of the above matrix as $\epsilon \to 0$

2. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of this matrix. How many eigenvalues are complex? Are there complex conjugate pairs?

Solution: Considering the characteristic polynomial we get

$$p_A(t) = -\lambda^3 + 1$$

And so the roots are $\lambda = 1, \frac{-1\pm\sqrt{3}i}{2}$.

The eigenvector for 1 is given by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the sum of the rows is constant.

Note the other 2 are complex conjugate and further are third roots of 1, ie $(\frac{-1\pm\sqrt{3}i}{2})^3 = 1$. Denote by $\omega = \frac{-1+\sqrt{3}i}{2}$. Then it is easy to check that the eigenvector of ω is $\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$ and by taking complex conjugates the eigenvector of ω is $\begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

1. Can you find an easy eigenvalue without computing the characteristic polynomial?
2. Compute all eigenvectors for the above easy eigenvalue
3. Can you use this to determine the remaining eigenvector and eigenvalue?

Solution: Note that clearly this matrix is singular and so has an eigenvalue 0. Further note all the columns are the same, so 0 has eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Since this is a symmetric matrix, the eigenvectors can be chossen to be orthogonal, so the remaining eigenvector has to be orthogonal to the above and hence has to be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, which after multiplying can be seen to have eigenvalue 3.

4. Let n be an odd number. Let A and $n \times n$ real matrix. Prove that this matrix always has a real eigenvalue.

Solution: Complex eigenvalues come in complex conjugate pairs also when counted in multiplicity. Thus we have an even number of non-real numbers. But in total, we have n eigenvalues when counted with multiplicities. Thus as n is odd we need to have at least one real eigenvalue.