Recitation 8. November 5

Focus: Algebraic and geometric multiplicity. Diagonalizability.

The eigenvalues of a square matrix A can be computed by finding the roots of the *characteristic polynomial* $det(A - \lambda I)$. The characteristic polynomial may have repeated roots, and the number of times a root λ_i appears is called the *algebraic multiplicity* of λ_i. The *geometric multiplicity* of an eigenvalue λ_i is the dimension of the nullspace of $(A - \lambda_i I)$, which is always at least 1. The geometric multiplicity of λ_i is at most the algebraic multiplicity of λ_i.

A square matrix A is *diagonalizable* if there exists a diagonal matrix D and invertible matrix S such that $SDS^{-1} = A$. The matrix A is diagonalizable if and only if the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue.

The matrix exponential is defined by $e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots + \frac{A^nt^n}{n!} + \cdots$. The formula $e^{SDS^{-1}t} = Se^{Dt}S^{-1}$ means that it is straightforward to calculate e^{At} whenever A is diagonalizable.

1. Which of the following matrices are diagonalizable? What are the algebraic and geometric multiplicities of the eigenvalues?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution:

2. Suppose that a 3×3 matrix A has an eigenvector $\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$ with eigenvalue 3. Name an eigenvalue and eigenvector of the matrix A^4.

Solution:

3. Suppose $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$.

   1. Find a diagonal matrix D and an invertible matrix S such that $SDS^{-1} = A$.
   2. Calculate A^4 and the matrix exponential e^{At}.

Solution:

4. Two matrices A and B are said to be *similar* if there is an invertible matrix S such that $A = SBS^{-1}$. Similar matrices have the same eigenvalues, and each of their eigenvalues have the same geometric and algebraic multiplicities. Construct two matrices A and B with the same characteristic polynomial, but which are not similar. Prove that if C and D are similar, then C^2 and D^2 are also similar.

Solution: