Recitation 8. November 5

Focus: Algebraic and geometric multiplicity. Diagonalizability.

The eigenvalues of a square matrix A can be computed by finding the roots of the *characteristic polynomial* $det(A - \lambda I)$. The characteristic polynomial may have repeated roots, and the number of times a root λ_i appears is called the *algebraic multiplicity* of λ_i. The *geometric multiplicity* of an eigenvalue λ_i is the dimension of the nullspace of $(A - \lambda_i I)$, which is always at least 1. The geometric multiplicity of λ_i is at most the algebraic multiplicity of λ_i.

A square matrix A is *diagonalizable* if there exists a diagonal matrix D and invertible matrix S such that $SDS^{-1} = A$. The matrix A is diagonalizable if and only if the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue.

The matrix exponential is defined by $e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots + \frac{A^nt^n}{n!} + \cdots$. The formula $e^{SDS^{-1}t} = Se^{Dt}S^{-1}$ means that it is straightforward to calculate e^{At} whenever A is diagonalizable.

1. Which of the following matrices are diagonalizable? What are the algebraic and geometric multiplicities of the eigenvalues?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution: The characteristic polynomial of A is $(1-\lambda)(3-\lambda) - (1)(0) = (1-\lambda)(3-\lambda)$. The eigenvalues are 1 and 3, each with algebraic multiplicity 1. Since geometric multiplicities are at least 1 and at most the algebraic multiplicities, we learn that the geometric multiplicities are also 1 and A is diagonalizable.

The characteristic polynomial of B is $(1-\lambda)(1-\lambda) - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2)$. The eigenvalues are 0 and 2, each with algebraic multiplicity 1. Again, each geometric multiplicity must be between 1 and the corresponding algebraic multiplicity, so the geometric multiplicities are both 1 and B is diagonalizable.

The characteristic polynomial of C is $(3-\lambda)^2$. The sole eigenvalue is 3, with algebraic multiplicity 2. Since $C - 3I$ is the 0 matrix, its nullspace is 2-dimensional, so the eigenvalue 3 has geometric multiplicity 2. It follows that C is diagonalizable (in fact, it is diagonal).

The characteristic polynomial of D is $(2-\lambda)^2$. The sole eigenvalue is 2, with algebraic multiplicity 2. The matrix $D - 2I$ has rank 1 and nullspace of dimension 1. Thus, the eigenvalue 2 has geometric multiplicity 1, and D is *not* diagonalizable. In the language of the lecture notes, D is a *Jordan block*.

2. Suppose that a 3×3 matrix A has an eigenvector $\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$ with eigenvalue 3. Name an eigenvalue and eigenvector of the matrix A^4.

Solution: Let v denote the vector $\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$. Then we know that $Av = 3v$. It follows that

$$A^4v = (AAAA)v = (AAA)(Av) = (AAA)(3v) = 3(AAAv) = 9AAv = 27Av = 81v.$$

Thus, v is an eigenvector of A^4 of eigenvalue 81.

3. Suppose $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$.

   1. Find a diagonal matrix D and an invertible matrix S such that $SDS^{-1} = A$.
   2. Calculate A^4 and the matrix exponential e^{At}.

Solution: For the first part, we can use

$$S = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that the diagonal entries of D are the eigenvalues of A. The columns of S are the eigenvectors of A.

To calculate A^4, we can use that

$$A^4 = (SDS^{-1})^4 = SD^4S^{-1} = S\begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix}S^{-1} = \begin{bmatrix} 81 & 32 \\ -81 & -16 \end{bmatrix}S^{-1} = \begin{bmatrix} 81 & 32 \\ -81 & -16 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -49 & -130 \\ 65 & 146 \end{bmatrix}.$$

We similarly calculate

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} e^{Dt} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

Note that

$$e^{Dt} = I + Dt + \frac{D^2t^2}{2} + \cdots = \begin{bmatrix} 1 + 3t + \frac{9t^2}{2} + \cdots & 0 \\ 0 & 1 + 2t + \frac{4t^2}{2} + \cdots \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^{3t} & 2e^{2t} \\ -e^{3t} & -e^{2t} \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^{3t} & 2e^{2t} - 2e^{3t} \\ e^{3t} - e^{2t} & 2e^{3t} - e^{2t} \end{bmatrix}.$$

4. Two matrices A and B are said to be *similar* if there is an invertible matrix S such that $A = SBS^{-1}$. Similar matrices have the same eigenvalues, and each of their eigenvalues have the same geometric and algebraic multiplicities. Construct two matrices A and B with the same characteristic polynomial, but which are not similar. Prove that if C and D are similar, then C^2 and D^2 are also similar.

Solution:

For matrices with the same characteristic polynomial that are not similar, we can use for example

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The characteristic polynomial of both of these matrices is $(2 - \lambda)^2$. However, the geometric multiplicities of the eigenvalue 2 are different, so the matrices are not similar. You should be aware that, even if matrices have the same characteristic polynomials and geometric multiplicities, that still does not guarantee that they are similar.

Next, suppose C and D are similar. That means we can find a matrix S such that $C = SDS^{-1}$. It follows that

$$C^2 = (SDS^{-1})^2 = (SDS^{-1})(SDS^{-1}) = SD(S^{-1}S)DS^{-1} = SDDS^{-1} = SD^2S^{-1},$$

so C^2 and D^2 are also similar.