Recitation 7. October 29

Focus: Determinants and inverses, eignevalues and eigenvectors

The **cofactors** C_{ij} of a square matrix A are given by $C_{ij} = (-1)^{i+j} det(M_{ij})$, where M_{ij} is the matrix given by removing the ith row and jth column of A. We can put these into the cofactor matrix X by $X_{ij} = C_{ji}$. We can compute the inverse using this matrix by $A^{-1} = \frac{1}{det(A)} X$. Using this we can also find a solution to $Ax = b$, by computing the entries of x, by $x_i = \frac{det(B_i)}{det(A)}$. Here B_i is given by replacing the ith column of A with the vector b

The ***eigenvectors*** of a square matrix A is a non-zero vector $\mathbf{v}$ that satisfies an equation $A\mathbf{v} = \lambda\mathbf{v}$ for some λ. In the above equation if you have an eigenvector the scalar λ is known as an **eigenvalue** of A. These can be computed by computing the zeroes of the equation $det(A - \lambda I)$.

1. Use the cofactor formula to invert the following matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution:

2. Use Cramer's rule to solve the following equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 1 \end{aligned} \qquad \begin{aligned} x + 3y - z &= 0 \\ x + y + 4z &= 0 \\ x + z &= 1 \end{aligned}$$

Solution:

3. a)Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $v \mapsto Av$. Can you find a basis $\mathbf{v}_1, \mathbf{v}_2$ with respect to which, ϕ is given by a diagonal matrix?
b) Do the same for the matrix

$$B = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

Solution:

4. a) Let A be an $n \times n$ matrix and let X be the cofactor matrix. Find a formula for $det(X)$ in terms of A
b) If A has an eigenvector $\mathbf{v}$, with eigenvalues λ, ie $A\mathbf{v} = \lambda\mathbf{v}$, show that $B = A - 7I$ also has $\mathbf{v}$ as an eigenvector. What is its eigenvalue?

Solution: