Recitation 7. October 29

Focus: Determinants and inverses, eignevalues and eigenvectors

The **cofactors** C_{ij} of a square matrix A are given by $C_{ij} = (-1)^{i+j} det(M_{ij})$, where M_{ij} is the matrix given by removing the ith row and jth column of A. We can put these into the cofactor matrix X by $X_{ij} = C_{ji}$. We can compute the inverse using this matrix by $A^{-1} = \frac{1}{det(A)} X$. Using this we can also find a solution to $Ax = b$, by computing the entries of x, by $x_i = \frac{det(B_i)}{det(A)}$. Here B_i is given by replacing the ith column of A with the vector b

The ***eigenvectors*** of a square matrix A is a non-zero vector $\mathbf{v}$ that satisfies an equation $A\mathbf{v} = \lambda \mathbf{v}$ for some λ. In the above equation if you have an eigenvector the scalar λ is known as an **eigenvalue** of A. These can be computed by computing the zeroes of the equation $det(A - \lambda I)$.

1. Use the cofactor formula to invert the following matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution: First we compute the matrix of cofactors for these 2 matrices. They are given by

$$\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

And the determinants of these matrices are given by 24 and 4 respectively, so the inverses are given by

$$\frac{1}{24}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

2. Use Cramer's rule to solve the following equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 1 \end{aligned} \qquad \begin{aligned} x + 3y - z &= 0 \\ x + y + 4z &= 0 \\ x + z &= 1 \end{aligned}$$

Solution: By Cramer's rule to compute x we replace the first column of the matrix with the solution vector and similarly for the other variable. So for the first equation

$$x = \frac{det(\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix})}{det(\begin{bmatrix} a & b \\ c & d \end{bmatrix})} = -\frac{b}{ad - bc}, y = \frac{det(\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix})}{det(\begin{bmatrix} a & b \\ c & d \end{bmatrix})} = \frac{a}{ad - bc}$$

For the second equation, first compute the determinant of the matrix $det(\begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}) = 11$. So we get the solution by Cramer's rule is given by

$$x = \frac{1}{11} det(\begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix} = \frac{13}{11}, y = \frac{1}{11} det(\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix} = -\frac{5}{11}, z = \frac{1}{11} det(\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -\frac{2}{11}$$

3. a)Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $v \mapsto Av$. Can you find a basis $\mathbf{v}_1, \mathbf{v}_2$ with respect to which, ϕ is given by a diagonal matrix?
b) Do the same for the matrix

$$B = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

Solution: a) First we compute the characteristic polynomial by $det(A - \lambda I) = (\lambda - 1)(\lambda + 1) - 3 = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$, so the eigenvalues of the matrix are given by $\lambda_1 = 2$ and $\lambda_2 = -2$. The eigenvectors for λ_1 are given by vectors in the nullspace of $A - 2I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$, so the eigenvector is given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The eigenvectors for λ_2 are given by vectors in the nullspace of $A + 2I = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$, so the eigenvector is given by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Note that with respect to the basis of eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ the linear transformation is given by the diagonal matrix $diag(2, -2)$
b) First we compute the characteristic polynomial by $det(A - \lambda I) = (\lambda - 2)(\lambda - 1)\lambda$, so the eigenvalues of the matrix are given by $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$. The eigenvectors for λ_1 are given by vectors in the nullspace of $A - 2I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$, so the eigenvector is given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The eigenvectors for λ_2 are given by vectors in the nullspace of $A - 1I = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}$, so the eigenvector is given by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The eigenvectors for λ_3 are given by vectors in the nullspace of $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so the eigenvector is given by $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Note that with respect to the basis of eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ the linear transformation is given by the diagonal matrix $diag(2, 1, 0)$

4. a) Let A be an $n \times n$ matrix and let X be the cofactor matrix. Find a formula for $det(X)$ in terms of A
b) If A has an eigenvector $\mathbf{v}$, with eigenvalues λ, ie $A\mathbf{v} = \lambda\mathbf{v}$, show that $B = A - 7I$ also has $\mathbf{v}$ as an eigenvector. What is its eigenvalue?

Solution: a) Note that $AX = det(A)I$. So taking determinant of this equation we get $det(A)det(X) = det(A)^n$. Hence it follows that $det(X) = det(A)^{n-1}$
b)$B\mathbf{v} = (A - 7I)\mathbf{v} = A\mathbf{v} - 7\mathbf{v} = (\lambda - 7)\mathbf{v}$, so indeed $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda - 7$