Recitation 6. October 22

Focus: linear transformations and matrix representations, determinants

A ***linear transformation*** is a map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$\phi(\boldsymbol{v} + \boldsymbol{w}) = \phi(\boldsymbol{v}) + \phi(\boldsymbol{w}) \qquad \text{and} \qquad \phi(\alpha \boldsymbol{v}) = \alpha\phi(\boldsymbol{v}).$$

A linear transformation ϕ can be expressed as a matrix A, with respect to given bases $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ and $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ of $\mathbb{R}^m$: the (i,j) entries a_{ij} of A are such that $\phi(\boldsymbol{v}_k) = a_{1k}\boldsymbol{w}_1 + \cdots + a_{mk}\boldsymbol{w}_m$.

The ***determinant*** of an $n \times n$ matrix A is the factor by which the linear map $\boldsymbol{v} \mapsto A\boldsymbol{v}$ scales volumes of regions in $\mathbb{R}^n$; it is denoted $\det A$.

1. Determine whether the following maps are linear. If so, find a matrix representation of the map in terms of the standard basis of $\mathbb{R}^3$, and then find a matrix representation in terms of the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (a) $\phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x+y+z \\ x^2+y^2+z^2 \\ 0 \end{bmatrix}$.

   (b) Let $\boldsymbol{a} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \in \mathbb{R}^3$, and define $\psi(\boldsymbol{v}) = (\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{a}$.

   (c) $\sigma\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x-y-z \\ x+2y \\ y-3z \end{bmatrix}$.

Solution:

2. Compute the determinant of

$$\begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & -2 \\ 1 & 3 & -1 & 2 \\ -1 & 3 & 0 & 5 \end{bmatrix}$$

by using row operations.

Solution:

3. Compute the determinant of

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 0 & -2 & 1 \\ 1 & 0 & -1 & 4 \end{bmatrix}$$

by doing a cofactor expansion along its second row.

Solution: