Recitation 6. October 22

Focus: linear transformations and matrix representations, determinants

A ***linear transformation*** is a map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$\phi(\boldsymbol{v} + \boldsymbol{w}) = \phi(\boldsymbol{v}) + \phi(\boldsymbol{w}) \quad \text{and} \quad \phi(\alpha \boldsymbol{v}) = \alpha\phi(\boldsymbol{v}).$$

A linear transformation ϕ can be expressed as a matrix A, with respect to given bases $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ of $\mathbb{R}^m$: the (i, j) entries a_{ij} of A are such that $\phi(\boldsymbol{v}_k) = a_{1k}\boldsymbol{w}_1 + \cdots + a_{mk}\boldsymbol{w}_m$.

The ***determinant*** of an $n \times n$ matrix A is the factor by which the linear map $\boldsymbol{v} \mapsto A\boldsymbol{v}$ scales volumes of regions in $\mathbb{R}^n$; it is denoted $\det A$.

1. Determine whether the following maps are linear. If so, find a matrix representation of the map in terms of the standard basis of $\mathbb{R}^3$, and then find a matrix representation in terms of the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (a) $\phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ x^2 + y^2 + z^2 \\ 0 \end{bmatrix}$.

   (b) Let $\boldsymbol{a} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \in \mathbb{R}^3$, and define $\psi(\boldsymbol{v}) = (\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{a}$.

   (c) $\sigma\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - y - z \\ x + 2y \\ y - 3z \end{bmatrix}$.

Solution:

(a) ϕ is not linear. For instance, $\phi\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ but $2\phi\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

(b) ψ is linear. Indeed, for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\psi(\alpha\boldsymbol{v} + \beta\boldsymbol{w}) = \big(\boldsymbol{a} \cdot (\alpha\boldsymbol{v} + \beta\boldsymbol{w})\big)\boldsymbol{a} = \big(\alpha(\boldsymbol{a} \cdot \boldsymbol{v}) + \beta(\boldsymbol{a} \cdot \boldsymbol{w})\big)\boldsymbol{a} = \alpha(\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{v} + \beta(\boldsymbol{a} \cdot \boldsymbol{w}) = \alpha\phi(\boldsymbol{v}) + \beta\phi(\boldsymbol{w}).$$

In terms of the standard basis, the matrix representation is

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) σ is linear. Indeed, clearly

$$\sigma\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so by the linearity of matrix multiplication σ is linear. Moreover, we see that $Y = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$ is therefore also the matrix representation of σ in terms of the standard basis.

To find the matrix representations of (b) and (c) in terms of the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

We can compute that

$$A^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}.$$

Then in terms of $\left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$, ψ is $A^{-1}XA$ and σ is $A^{-1}YA$. Explicitly,

$$A^{-1}XA = \frac{1}{4}\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

and

$$A^{-1}YA = \frac{1}{2}\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\ 1 & 2 & 0\\ 0 & 1 & -3\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ -1 & 2 & 1\\ -1 & 4 & -3\end{bmatrix} = \begin{bmatrix}2 & -3 & 1\\ 1 & -1 & 2\\ 0 & 3 & -1\end{bmatrix}.$$

2. Compute the determinant of

$$M = \begin{bmatrix}0 & 0 & 2 & -1\\ 0 & 0 & -4 & -2\\ 1 & 3 & -1 & 2\\ -1 & 3 & 0 & 5\end{bmatrix}$$

by using row operations.

Solution: We first swap the first and third rows, and then the second and fourth rows to arrive at the matrix

$$M' = \begin{bmatrix}1 & 3 & -1 & 2\\ -1 & 3 & 0 & 5\\ 0 & 0 & 2 & -1\\ 0 & 0 & -4 & -2\end{bmatrix}.$$

Therefore $\det M = (-1)^2 \det M' = \det M'$. We now perform elimination operations on M':

$$\begin{bmatrix}1 & 3 & -1 & 2\\ -1 & 3 & 0 & 5\\ 0 & 0 & 2 & -1\\ 0 & 0 & -4 & -2\end{bmatrix} \to \begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 6 & -1 & 7\\ 0 & 0 & 2 & -1\\ 0 & 0 & -4 & -2\end{bmatrix} \to \begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 6 & -1 & 7\\ 0 & 0 & 2 & -1\\ 0 & 0 & 0 & -4\end{bmatrix},$$

which shows $\det(M') = 1 \cdot 6 \cdot 2 \cdot (-4) = -48$. Thus $\det M = -48$. Note that

$$\det M = \det\left(\begin{bmatrix}1 & 3\\ -1 & 3\end{bmatrix}\right)\det\left(\begin{bmatrix}2 & -1\\ -4 & -2\end{bmatrix}\right).$$

Indeed, it is true in general (and can be seen by row operation, for instance) that if a matrix is of the form $\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}$, then its determinant is $\det(A)\det(C)$.

3. Compute the determinant of

$$B = \begin{bmatrix}1 & 2 & -1 & 0\\ 3 & -2 & 0 & 5\\ -2 & 0 & -2 & 1\\ 1 & 0 & -1 & 4\end{bmatrix}$$

by doing a cofactor expansion along its second row.

Solution: We have that $\det B$ equals

$$(-1)^{2+1}\cdot 3\det\left(\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 1\\ 0 & -1 & 4\end{bmatrix}\right)+(-1)^{2+2}\cdot(-2)\det\left(\begin{bmatrix}1 & -1 & 0\\ -2 & -2 & 1\\ 1 & -1 & 4\end{bmatrix}\right)+(-1)^{2+4}\cdot 5\det\left(\begin{bmatrix}1 & 2 & -1\\ -2 & 0 & -2\\ 1 & 0 & -1\end{bmatrix}\right)=$$

$$-3(2((-2)(4)-(-1)))-2\big(((-2)(4)-(-1))+((-2)(4)-1)\big)+5((-1)(2)((-2)(-1)-(-2)))=42+32-40=34.$$

So, $\det B = 34$.