Recitation 3. September 24

Focus: vector spaces and subspaces, the column space and null space of a matrix

A ***vector space*** is a set V in which you may add and scale vectors; a ***subspace*** of V is a subset of V which is closed under addition of vectors and scalar multiplication.

Let A be an $m \times n$ matrix. The ***column space*** $C(A)$ of A is the span of its columns; it is a subspace of $\mathbb{R}^m$. The ***null space*** $N(A)$ of A the set of vectors $\boldsymbol{v}$ such that $A\boldsymbol{v} = \mathbf{0}$; it is a subspace of $\mathbb{R}^n$.

1. Determine whether the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$:

   (a) The set of vectors of the form $\begin{bmatrix} 1 \\ -1 \\ a \end{bmatrix}$, where a is some real number.

   (b) The set $\{\mathbf{0}\}$ consisting of only the zero vector.

   (c) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the equation $4x - 3y + 2z = 0$.

Solution:

2. Let

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -7 & -1 \\ -3 & 12 & 3 \end{bmatrix}.$$

   Determine the space W of vectors $\boldsymbol{b}$ such that $A\boldsymbol{v} = \boldsymbol{b}$ has a solution. Find a lower triangular 3×3 matrix whose column space is W.

Solution:

3. Use Gauss-Jordan elimination to compute the null space $N(X)$ of the matrix

$$X = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 0 & -2 & 1 \end{bmatrix}.$$

Solution:

4. Let B be an $m \times n$ matrix. Show that if $\boldsymbol{v} \in C(B^T)$, then $\boldsymbol{v} \cdot \boldsymbol{u} = 0$ for any $\boldsymbol{u} \in N(B)$.

Solution: