Recitation 2. September 17

Focus: multiplying matrices (and taking inverses), $A = LU$, $A = LDU$ and $PA = LU$ factorizations, transposes, symmetric matrices

The LU factorization of a square matrix A is the unique way of writing it:

$$\boxed{A = LU}$$

where L is a lower-diagonal matrix with 1 on the diagonal and U is an upper-diagonal matrix. This works for almost all matrices A. Even for those for which this doesn't work, you can always write $PA = LU$ for a suitable permutation matrix P.

1. Show that for any matrix A, the square matrix $S = A^T A$ is symmetric. For any vector $\boldsymbol{v}$, show that:

$$\boldsymbol{v}^T \underbrace{A^T A}_{S} \boldsymbol{v} \tag{1}$$

is a (1×1 matrix whose only entry is a) non-negative number.

Solution: S being symmetric boils down to the fact that $S^T = (A^T A)^T = A^T (A^T)^T = A^T A = S$. As for (1), if we consider the vector:

$$A\boldsymbol{v} = \boldsymbol{w} = \begin{bmatrix} w_1 \\ \dots \\ w_m \end{bmatrix}$$

then:

$$\boldsymbol{v}^T A^T A\boldsymbol{v} = (\boldsymbol{v}^T A^T)(A\boldsymbol{v}) = (A\boldsymbol{v})^T (A\boldsymbol{v}) = \boldsymbol{w}^T \boldsymbol{w} = w_1^2 + \dots + w_m^2 \geq 0$$

This non-negativity will play an important role in a few weeks.

2. Compute the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 6 & -1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

by Gauss-Jordan elimination on the augmented matrix $\begin{bmatrix} A | I \end{bmatrix}$.

Solution: The augmented matrix is:

$$\left[\begin{array}{cccccc} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ \boxed{3} & 1 & 2 & 0 & 1 & 0 \\ \boxed{2} & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

(pivots are boxed) The first step in Gauss-Jordan elimination is to subtract 3 times the first row from the second row and 2 times the first row from the third row:

$$\left[\begin{array}{cccccc} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{-17} & 5 & -3 & 1 & 0 \\ 0 & \boxed{-10} & 3 & -2 & 0 & 1 \end{array}\right]$$

Then we subtract $\frac{10}{17}$ times the second row from the third row:

$$\left[\begin{array}{cccccc} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{-17} & 5 & -3 & 1 & 0 \\ 0 & 0 & \boxed{\dfrac{1}{17}} & -\frac{4}{17} & -\frac{10}{17} & 1 \end{array}\right]$$

The next step is to make all pivots 1, by dividing the second row by -17 and multiplying the third row by 17:

$$\left[\begin{array}{cccccc} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -\frac{5}{17} & \frac{3}{17} & -\frac{1}{17} & 0 \\ 0 & 0 & \boxed{1} & -4 & -10 & 17 \end{array}\right]$$

To complete Gauss-Jordan elimination, we need to make the entries above the pivots 0. To do so, we first add $\frac{5}{17}$ times the third row to the second row:

$$\begin{bmatrix} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & -1 & -3 & 5 \\ 0 & 0 & \boxed{1} & -4 & -10 & 17 \end{bmatrix}$$

Then we add -6 times the second row to the first row and 1 times the third row to the first row:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 3 & 8 & -13 \\ 0 & \boxed{1} & 0 & -1 & -3 & 5 \\ 0 & 0 & \boxed{1} & -4 & -10 & 17 \end{bmatrix}$$

Thus, the inverse is:

$$A^{-1} = \begin{bmatrix} 3 & 8 & -13 \\ -1 & -3 & 5 \\ -4 & -10 & 17 \end{bmatrix}$$

3. Compute the $PA = LDU$ factorization of the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

Solution: There are two choices for the 2×2 permutation matrix P:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first one will not work, since $IA = A$ does not have an LU factorization (this is because Gaussian elimination will not work on the matrix A without a row exchange). Therefore, let us exchange the rows of A, which is achieved by multiplying with the second permutation matrix above:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

This matrix is already in row echelon for, so we conclude that $PA = LU$ with:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

4. Write down the 4×4 matrices corresponding to the permutation $\{2, 1, 4, 3\}$ and $\{2, 3, 4, 1\}$ and compute their product. Is the product also a permutation matrix, and if so, to which permutation does it correspond?

Solution: The permutation matrices corresponding to the two permutations are:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The product of these matrices is:

$$P_1 P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the permutation matrix corresponding to the permutation $\{3, 2, 1, 4\}$. In general, the product of the permutation matrices corresponding to permutation $\{\sigma(1), ..., \sigma(n)\}$, $\{\sigma'(1), ..., \sigma'(n)\}$ will be the permutation matrix corresponding to the permutation $\sigma'(\sigma(1)), ..., \sigma'(\sigma(n))$.