Recitation 1. September 10

Focus: rules of matrix multiplication; Gauss-Jordan elimination is LU factorization.

Remark. The most basic rule that you should remember: **row column**. It shows the order in which you write or compute, e.g.:

- The first index denotes the row, the second number the column.
- You multiply a row by a column to get a number.
- An $n \times m$ matrix has n rows and m columns.

Remark. The formula left matrix multiplication corresponds to row operations explains the mathemagic behind Gauss-Jordan elimination. More precisely, performing row operations on a matrix A is the same as doing LA for some other matrix L.

LU factorization of a matrix A is a way of writing A as a product of two matrices $\boxed{A = LU}$, where L is a lower-diagonal matrix with units on the diagonal and U is an upper-diagonal matrix.

1. *Rules of matrix multiplication. (Section 2.4 of Strang.)* Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}, \; C = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \; D = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Which of these matrix operations are allowed?

a) AB
b) $(A+B)C$
c) $C(A+B)$
d) AD
e) DA
f) CAD

Solution: In order to multiply two matrices, number of columns in the first should be equal to number of rows in the second.

a) AB not allowed: we cannot multiply a 2×3 matrix by a 2×3 matrix.

b) $(A+B)C$ not allowed.

c) $C(A+B) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -2 \\ -4 & -4 & -8 \end{pmatrix}$.

d) $AD = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$.

e) DA not allowed.

f) $CAD = C(AD) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} -14 \\ -18 \end{pmatrix}$.

2. *Binomial formula for matrices.* Show that $(A+B)^2$ is different from $A^2+2AB+B^2$ when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}.$$

Write down the correct rule: $(A+B)^2 = A^2 + \cdots + B^2$.

Solution:

- $(A+B)^2 = \begin{bmatrix}2 & 2\\ 3 & 0\end{bmatrix}^2 = \begin{bmatrix}10 & 4\\ 6 & 6\end{bmatrix}$;
- $A^2+2AB+B^2 = \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} + 2\begin{bmatrix}7 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix} = \begin{bmatrix}16 & 2\\ 3 & 0\end{bmatrix}$.
- The correct rule is $(A+B)^2 = A^2+AB+BA+B^2$.

3. Consider the following matrices:

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix},\ B=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},\ C=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ D=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}.$$

How is each row of BA, CA, DA related to the rows of A?

Solution:

- The first row of BA is twice the first row of A, and the second is minus the second row of A.
- The first row of CA is the second row of A, while the second row is zero.
- The first row of DA is the second row of A and the second row of DA is minus the second row of A.

So you can see that multiplying a matrix A by another matrix on the left performs row operations. Similarly, right multiplication performs column operations.

4. *LU factorization = Gaussian elimination.* Solve the system of linear equations using LU factorization:

$$\begin{cases} x+2y+3z=1,\\ \quad\ \ y+\ z=2,\\ 3x+\ y-\ z=3.\end{cases}$$

Solution: We start by writing the system of linear equations in the matrix form:

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 3 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}.$$

Now we perform Gauss-Jordan elimination on the augmented matrix:

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & 1 & 1 & 2\\ 3 & 1 & -1 & 3\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & 1 & 1 & 2\\ 0 & -5 & -10 & 0\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & 1 & 1 & 2\\ 0 & 0 & -5 & 10\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & -2\end{array}\right].$$

So we can deduce that $z=-2$, and by back substitution we get $y=4$ and $x=-1$.

5. *Not all matrices can be written in LU form. (You will deal with this kind of situation when discussing $PA = LU$ factorizations on Friday.)* Show directly why this matrix equation is impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}.$$

Solution: Let us multiply out the right hand side of the equation:

$$\begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} d & e \\ dl & le + f \end{bmatrix}.$$

Then equate both sides:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} d & e \\ dl & le + f \end{bmatrix}.$$

Now try to solve this equation. First, we observe that $d = 0$. But then, $2 = dl = 0 \cdot l = 0$, a contradiction. So we cannot find numbers d, e, f, l such that the formula above holds.