Solution pset 9

1a) Yes. A = not invertible if it has a zero eigenvalue.

+ve def matrices do not have zero eigenvalues.

b) TEST 4 says to check $\underline{x}^T A^T A \underline{x} \geq 0, \underline{x} \neq 0$

$$\underline{x}^T A^T A \underline{x} = (A\underline{x})^T A \underline{x} = \underbrace{\|A\underline{x}\|}_{\text{length squared}}{}^2 \geq 0 \qquad \underline{x} \neq 0$$

c) From part c)

For $\underline{x} \neq 0$, $\underline{x}^T A^T A \underline{x} = \|A\underline{x}\|^2 > 0$ if $A\underline{x} \neq \underline{0}$

$\therefore$ $N(A)$ is empty implies $A^T A$ is +ve def$^{\underline{n}}$.

d) This is great! For least squares we solve

$$\underbrace{A^T A}_{} \underline{x} = A^T \underline{b}$$

This is invertible if $N(A) = \{\underline{0}\}$

or cols of A are linearly independent.

2) ① Calculate eigenvalues of A^TA:

$$A^TA = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

eigs of A^TA are 6 and 1

eigenvectors ~~of~~ are

$$A^TA \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad A^TA \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

② Calculate eigenvalues of AA^T.

$$AA^T = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

eigs are 6, ~~&~~ 1, and 0.

eigenvectors:

$$AA^T \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \quad AA^T \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$AA^T \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} -2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & 1/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \end{bmatrix}$$

③ Check signs:

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} \stackrel{!}{=} \underbrace{\begin{bmatrix} -2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & 1/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ +1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{V^T}$$

maybe have to put minus sign here ~~of~~ or here ~~and~~ or here

~~$U = [\,2/\sqrt{3}$~~

This is getting complicated: Let me check that (as an alternative):

$$A\underline{v}_1 = \sigma_1 \underline{u}_1$$

$$A\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -5/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = \sqrt{6}\begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix} = \sqrt{6}\left(-\underline{u}_1\right)$$

$$A\underline{v}_2 = \sigma_2 \underline{u}_2$$

$$A\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} = \underline{u}_2$$

Therefore, first col of U needs a minus.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ -5/\sqrt{30} & 0 & 1/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{V^T}$$

3a) $T(\cancel{3}\,ap) = 3ap(x) + 5 \neq aT(p)$

NOT LINEAR.

b) $T(ap) = axp(x) + 7ap'(x) = aT(p)$

$T(p+q) = xp + xq + 7(p'(x) + q'(x))$

$= T(p) + T(q)$

LINEAR

c) $T(ap) = a^3 p'(x) p(x)^2 \neq aT(p)$

NOT LINEAR

d) $T(aA) = (aA)^T = aA^T = aT(A)$

$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$

LINEAR

e) $T(aA) = (aA)^{-1} = \frac{1}{a} A^{-1} \neq aT(A)$

NOT LINEAR

4 a) It costs $O(n^2)$ to multiply $B\underline{x}_k$

and costs $O(n)$ to inverse diagonal matrix.

Total is $\underline{\underline{O(n^2)}}$. " means ~~some multiple~~ grows like ~~of~~ n^2 "

b) Then $\underline{v}$ satisfies

$$D\underline{v} = -B\underline{v} + \underline{b}$$

and so $(D+B)\underline{v} = A\underline{v} = \underline{b}$

c)

$$
\begin{aligned}
x_{k+1} &= -D^{-1}B\underline{x}_k + D^{-1}\underline{b} \\
&= -D^{-1}B\left(D^{-1}B\underline{x}_{k-1} + D^{-1}\underline{b}\right) + D^{-1}\underline{b} \\
&= -(D^{-1}B)^2 \underline{x}_{k-1} \cancel{+} - D^{-1}BD^{-1}\underline{b} + D^{-1}\underline{b}.
\end{aligned}
$$

d)

$$
\begin{aligned}
\underline{x}_{k+1} &= (D^{-1}B)^3 \underline{x}_{k-2} + (D^{-1}B)^2 D^{-1}\underline{b} - D^{-1}BD^{-1}\underline{b} + D^{-1}\underline{b} \\
&= -(D^{-1}B)^4 \underline{x}_{k-3} \cancel{\mp} (D^{-1}B)^3 D^{-1}b + (D^{-1}B)^2 D^{-1}\underline{b} - D^{-1}BD^{-1}\underline{b} \\
&\qquad + D^{-1}\underline{b}. \\
&= \dots = (-1)^{k+1}(D^{-1}B)^{k+1}\underline{x}_0 + \sum_{j=0}^{k}(-1)^j (D^{-1}B)^j D^{-1}\underline{b} \\
&= \sum_{j=0}^{k+1}(-1)^j (D^{-1}B)^j D^{-1}\underline{b} \qquad \left(\text{as } \underline{x}_0 = D^{-1}\underline{b}\right)
\end{aligned}
$$

9e)

$$\underline{x}_{k+1} = \sum_{j=0}^{k+1} (-1)^j (D^{-1}B)^j D^{-1}\underline{b}$$

If $|\lambda(D^{-1}B)| < 1$ for all eigenvalues, then this sum converges, as $k \longrightarrow \infty$ (each term gets smaller).