Course 18.06: Problem Set 9

Due 4PM, Thursday 3rd December 2015, in the boxes at E17-131.

This homework has 4 questions to hand-in. Write down all details of your solutions, not just the answers. Show your reasoning. Please staple the pages together and clearly PRINT your name, recitation section, and the name of your recitation instructor on the first page of the problem set. You know the collaboration rules by now. This homework also has an online self-graded part.

Problems 1–4

1. [10 pts] (a) Are all positive definite matrices invertible?
(b) Let A be an $m \times n$ real matrix where $m > n$. Show that $A^T A$ is positive semi-definite, i.e., it is symmetric and all eigenvalues are ≥ 0.
(c) Give a condition on A to ensure that $A^T A$ is a positive definite matrix.
(d) Use part (a) and (c) to explain why this is great news for least squares fitting.

2. [10pts] (a) Compute the singular value decomposition of

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 1 \end{bmatrix}.$$

(b) If $\underline{v}$ is a vector so that $\|v\| = 1$, then what are lower and upper bounds on $\|A\underline{v}\|$?

3. [15 pts] Let us consider operations on the space of polynomials $p(x)$ of degree at most n. Are the following linear transforms? Give reasons.
(a) $T(p) = 3p(x) + 5$,
(b) $T(p) = xp(x) + 7p'(x)$,
(c) $T(p) = p'(x)p(x)^2$,
(d) Is the transform $T(A) = A^T$ a linear transform on real matrices?
(e) Is the transform $T(A) = A^{-1}$ a linear transform on invertible matrices?

4. [15 pts] Let's solve $A\underline{x} = \underline{b}$ by an iteration (maybe A is too large and sparse, and elimination is too expensive). To tackle this problem, let's suppose that the diagonal entries of A are nonzero and are much larger than the off-diagonal entries. We write

$$A = D + B,$$

where D is a diagonal matrix formed by extracting the diagonal entries of A and B are the leftover off-diagonal entries. We expect $\underline{x} \approx D^{-1}\underline{b}$ and will do an iteration to improve that first guess.

We use the iteration: $\underline{x}_0 = D^{-1}\underline{b}$,

for $k = 0, 1, 2, \ldots,$

Solve $D\underline{x}_{k+1} = -B\underline{x}_k + \underline{b}$

end

(a) What is the cost of one step of the iteration?
(b) If the iteration converges, i.e., $\underline{v} = \lim_{k\to\infty} \underline{x}_k$, then explain why $A\underline{v} = \underline{b}$.
(c) Note that $\underline{x}_{k+1} = -D^{-1}B\underline{x}_k + D^{-1}\underline{b}$. Show that

$$\underline{x}_{k+1} = (D^{-1}B)^2\underline{x}_{k-1} - D^{-1}BD^{-1}\underline{b} + D^{-1}\underline{b}$$

(d) Keep going and show that

$$\underline{x}_{k+1} = \sum_{j=0}^{k+1} (-1)^j (D^{-1}B)^j D^{-1}\underline{b}$$

(It is sufficient here to plug in expressions a few times and identify the repeating patterns. Your mathy friends will do this by induction, but you can get full credit without being so rigorous.)
(e) Use part (d), and your knowledge of eigenvalues and powers of matrices, to give a condition that ensures that the iteration converges as $k \to \infty$. (Hint: Again you can get full credit here without being rigorous.)