Solutions pset 8

1a) TRUE

$$\det(A-\lambda I) = \det(SBS^{-1}-\lambda I) = \det(S(B-\lambda I)S^{-1}) = \det(B-\lambda I).$$

b) TRUE

$$A - 5I = SBS^{-1} - 5I = S(B-5I)S^{-1}.$$

c) TRUE

$$A = SBS^{-1}$$

so $A^T = (S^{-1})^T B^T S^T = (S^T)^{-1} B^T S^T$.

d) ~~FALSE~~ TRUE so $A = SBS^{-1}$, $AS = SB$,

$$S(BA)S^{-1} = (SB)AS^{-1} = ASAS^{-1} = AB.$$

e) $e^{At} = e^{SBS^{-1}t} = Se^{Bt}S^{-1}.$

2)

$$e^{At} = \begin{bmatrix} e^{3t} & \\ & e^{-t} \end{bmatrix}$$

$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, $\det(A-\lambda I)=0 \Rightarrow \lambda_1 = 4,\ \lambda_2 = -2$

$A\begin{bmatrix} 5 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

so $A = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$\therefore \quad e^{At} = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & \\ & e^{-2t} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}^{-1}.$

For $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, we have $A^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} t^k/k! & \sum_{k=0}^{\infty} 2k\, t^k/k! \\ 0 & \sum_{k=0}^{\infty} t^k/k! \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}$$

3 a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$

c) Suppose ~~Ax~~ $A\underline{x} = \lambda \underline{x}$ for $\underline{x} \neq 0$, then λ is an eigenvalue of A. I will show that $\bar{\lambda}$ is eigenvalue of A^*

$$\det(A^* - \bar{\lambda} I) = \det(\bar{A}^T - \bar{\lambda} I) = \overline{\det(A^T - \lambda I)} = \overline{\det((A - \lambda I)^T)} = \overline{\det(A - \lambda I)} = 0$$

d) $A\underline{x} = \lambda \underline{x}$ and $A = A^*$

$$\lambda \underline{x}^* \underline{x} = \underline{x}^*(\lambda \underline{x}) = \underline{x}^*(A\underline{x}) = (\underline{x}^* A)\underline{x}$$
$$= (A^* \underline{x})^* \underline{x}$$
$$= (A\underline{x})^* \underline{x}$$
$$= (\lambda \underline{x})^* \underline{x}$$
$$= \bar{\lambda}\, \underline{x}^* \underline{x}$$

Since $\underline{x} \neq 0$, $\underline{x}^*\underline{x} \neq 0$, and hence $\lambda = \bar{\lambda}$, (λ is real).

e)
$$A = \begin{bmatrix} i & 2 \\ 2+i & 1-2i \end{bmatrix}$$

let $B = \begin{bmatrix} 0 & 2-\frac{1}{2}i \\ 2+\frac{1}{2}i & 1 \end{bmatrix}$ and $C = \begin{bmatrix} +1 & +1/2 \\ +1/2 & -2 \end{bmatrix}$.

f) Yes. In general let
$B = (A + A^*)/2$ and $C = -\frac{i}{2}(A - A^*)$.

4) a) Every polynomial of degree n can be writen as a linear comb. of $P_0, \dots, P_n$.

b) Need to show that $P_0, \dots, P_n$ are linearly independent.
Since they are orthogonal, they must be linearly independent

c)

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + \dots + \dots$$

so $$\int_{-1}^{1} f(x) P_K(x) = \sum_{j=0}^{\infty} a_j \int_{-1}^{1} P_j(x) P_K(x)\,dx$$

$$= a_K \int_{-1}^{1} P_K(x) P_K(x)\,dx$$

$$= \frac{2}{2K+1} a_K$$

$\therefore$

$$a_K = \frac{2K+1}{2} \int_{-1}^{1} f(x) P_K(x)\,dx.$$