Course 18.06: Problem Set 8

Due 4PM, Thursday 19th November 2015, in the boxes at E17-131.

This homework has 4 questions to hand-in. Write down all details of your solutions, not just the answers. Show your reasoning. Please staple the pages together and clearly PRINT your name, recitation section, and the name of your recitation instructor on the first page of the problem set. You know the collaboration rules by now. This homework also has an online self-graded part.

Problems 1–4

1. [10 pts] We say that A and B are similar matrices if $A = SBS^{-1}$ for some invertible matrix S. Are the following TRUE or FALSE? Give a reason:
(a) If A and B are similar, then A and B have the same eigenvalues,
(b) If A and B are similar, then $A - 5I$ and $B - 5I$ are similar,
(c) If A and B are similar, then A^T and B^T are similar,
(d) If A and B are similar, then AB and BA are similar,
(e) If A and B are similar, then e^{At} and e^{Bt} are similar.

2. [10pts] For the following matrices compute e^{At}:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

3. [15 pts] (a) Give a 2×2 real matrix with eigenvalues that are not real.
(b) Is there a 2×2 complex (but not real) matrix with real eigenvalues?
(c) Show that if $A\underline{x} = \lambda\underline{x}$, then $A^*\underline{x} = \overline{\lambda}\underline{x}$, where A^* is the conjugate transpose of A and $\overline{\lambda}$ is the complex conjugate of λ.
(d) Explain why eigenvalues of Hermitian matrices are always real.
(e) Write the complex matrix

$$A = \begin{bmatrix} i & 2 \\ 2+i & 1-2i \end{bmatrix}$$

as a sum $A = B + iC$, where B and C are Hermitian.
(f) Can every $n \times n$ complex matrix A be written as $A = B + iC$, where B and C are Hermitian?

4. [15 pts] Let P_0, P_1, …, P_n be the first $n+1$ Legendre polynomials and let $\mathcal{P}_n$ be the set of polynomials of degree at most n. The Legendre polynomials satisfy:

- P_k is a polynomial of degree k,
- The first three are: $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$.
- The polynomials are orthogonal on $[-1, 1]$,

$$\int_{-1}^{1} P_j(x)P_k(x)dx = \begin{cases} \frac{2}{2k+1}, & j = k, \\ 0, & j \neq k. \end{cases}$$

(a) Why does P_0, P_1, …, P_n span $\mathcal{P}_n$?
(b) Show that $\{P_0, P_1, \ldots, P_n\}$ is a basis for $\mathcal{P}_n$.

(c) Let $f(x)$ be a continuous function on $[-1, 1]$. Suppose that

$$f(x) = \sum_{j=0}^{\infty} a_k P_k(x), \qquad x \in [-1, 1].$$

Give a formula for a_k involving integrals. (Think about vectors.)

(Almost everything we have done in 18.06 for matrices has a continuous analogue for functions. This is used all the time in 18.03, signal processing, seismic imaging, and way beyond.)