pset 7

1a) $\det(B^{-1}AB - \lambda I) = \det(B^{-1}(A-\lambda I)B) = \det(A-\lambda I)$

(b) Consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) let λ_1 and λ_2 be different and $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$.

then suppose $c_1 v_1 + c_2 v_2 = 0$

$$\Rightarrow \quad A(c_1 v_1 + c_2 v_2) = c_1 A v_1 + c_2 A v_2 \cancel{=0}$$
$$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \quad ①$$

and $\quad c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0 \quad ②$

② − ① : $\cancel{c_2(\lambda_2 - \lambda_1)}$ $c_2(\lambda_1 - \lambda_2)\underline{v}_2 = 0$

Since $\underline{v}_2 \neq 0$, $\lambda_1 \neq \lambda_2 \Rightarrow c_2 = 0$.

$\therefore \quad c_1 v_1 = 0 \quad \Rightarrow \quad c_1 = 0.$

2. $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 + 1$

$= (\lambda - (1+i))(\lambda - (1-i))$

$\lambda_1 = (1+i)\,, \quad \lambda_2 = (1-i)$

$B(t) = \begin{bmatrix} (1-t)(1+i) & -t \\ -t & (1-t)(1-i)+t \end{bmatrix} \qquad D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$

The eigenvalues of $B(t)$ are the roots of

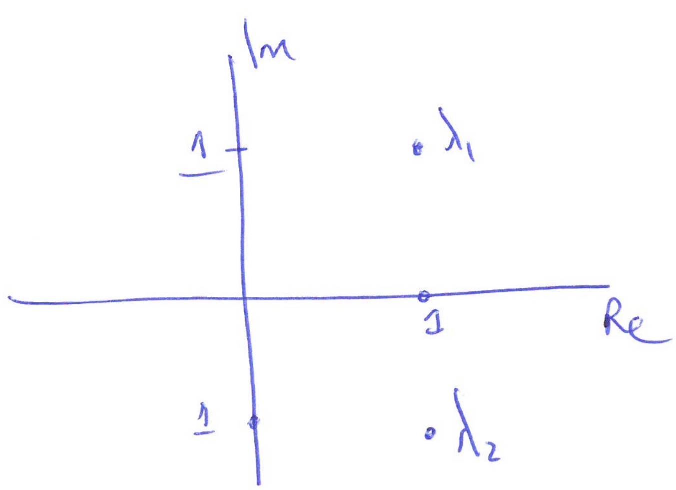
$\det(B(t) - \lambda I) = \lambda^2 - 2\lambda - 2t + 2$

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4 - 4(2-2t)}}{2}$$

$= 1 \pm \sqrt{1-(2-2t)}$

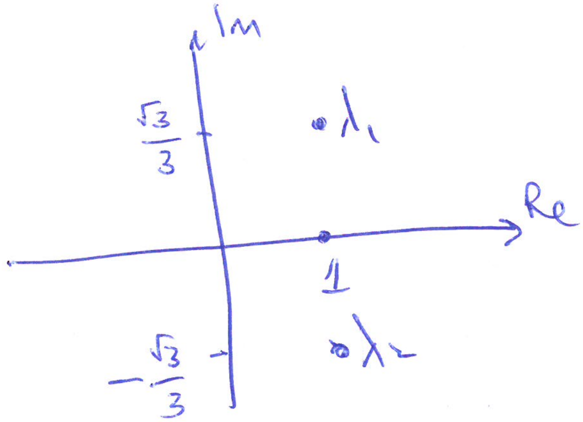
$= 1 \pm \sqrt{2t-1}$

<u>At $t=0$</u> : $B(0) = D$

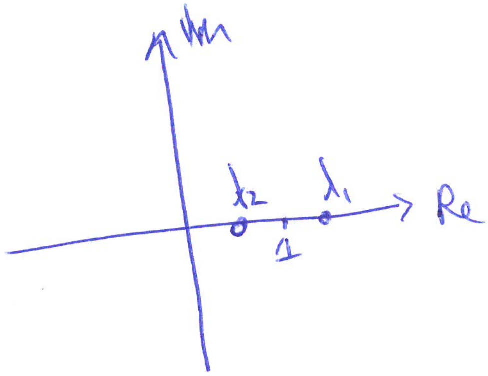


At $t = 1/3$: $B(1/3) = 2/3\, D + \frac{1}{3} A$.

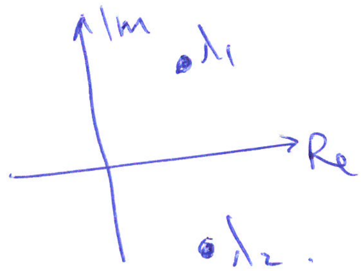
$$\lambda_1 = 1 + \frac{\sqrt{3}}{3} i$$
$$\lambda_2 = 1 - \frac{\sqrt{3}}{3} i$$



At $t = 2/3$: $\lambda_1 = 1 + \frac{\sqrt{3}}{3}$, $\lambda_2 = 1 - \frac{\sqrt{3}}{3}$



At $t = 1$: $\cancel{A=}$ $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$



3a) ~~det(A~~ let $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} \cancel{= (\lambda+3)}$$

$$= (\lambda+3)(\lambda+2) - 2$$

$$= (\lambda+1)(\lambda+4)$$

$$\lambda_1 = -1, \quad \lambda_2 = -4.$$

<u>For $\lambda_1 = -1$</u>:

$$\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$\text{so } v_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

<u>For $\lambda_2 = -4$</u>

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{so } v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Let $S = \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix}$, then

$$A = S \begin{bmatrix} -1 & \\ & -4 \end{bmatrix} S^{-1}.$$

So

$$\frac{du}{dt} = S \begin{bmatrix} -1 & \\ & -4 \end{bmatrix} S^{-1} \underline{u}$$

let $\underline{w} = S^{-1}\underline{u}$, then

$$\frac{d\underline{w}}{dt} = \frac{d(S^{-1}\underline{u})}{dt} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \underline{w}$$

and $\underline{w}(0) = S^{-1}\underline{u}(0) = S^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1/3 \end{pmatrix}$

$$\therefore \quad \underline{w}(t) = \begin{pmatrix} 4/3\, e^{-t} \\ \frac{1}{3} e^{-4t} \end{pmatrix}$$

and $\underline{u}(t) = S\underline{w}(t) = \begin{pmatrix} 2/3\, e^{-t} + 1/3\, e^{-4t} \\ 4/3\, e^{-t} - 1/3\, e^{-4t} \end{pmatrix}$

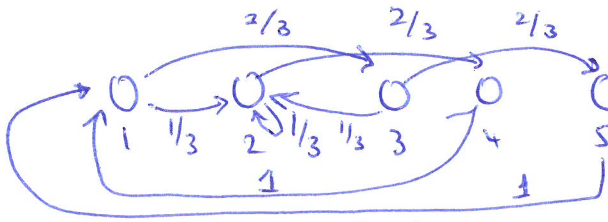
(b) Now ~~A is~~ we have

$A + 2I$ so eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$

$\therefore$ solution explodes like e^t.

Yes. If $\underline{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then no component in blowing up solution

4 a) The associated Markov matrix is



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 \end{bmatrix}$$

Using MATLAB

$\cancel{A}$ $\begin{bmatrix} 0.6149 \\ 0.5124 \\ 0.4099 \\ 0.3416 \\ 0.2733 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue 1.

The frog wins because $0.3416 > 0.2733$.