# Course 18.06: Problem Set 7

Due 4PM, Thursday 12th November 2015, in the boxes at E17-131.

This homework has 4 questions to hand-in. Write down all details of your solutions, not just the answers. Show your reasoning. Please staple the pages together and clearly PRINT your name, recitation section, and the name of your recitation instructor on the first page of the problem set. You know the collaboration rules by now. This homework also has an online self-graded part.

## Problems 1–4

1. [15 pts] (a) Let $B$ be an invertible matrix, show that $C = B^{-1}AB$ has the same eigenvalues as $A$. We say that $A$ and $C$ are *similar* matrices.

(b) Show that the converse is false, i.e., if $A$ and $C$ have the same eigenvalues then there may not be a matrix $B$ such that $C = B^{-1}AB$.

(c) Show that two eigenvectors corresponding to distinct eigenvalues are linearly independent.

2. [10 pts] Let $A$ be the following matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Calculate the eigenvalues $\lambda_1$ and $\lambda_2$ of $A$. Set $D = diag(\lambda_1, \lambda_2)$ to be a diagonal matrix of eigenvalues and let $B(t) = (1-t)D + tA$, $0 \leq t \leq 1$. By considering $\det(B(t) - xI)$, calculate expressions for the eigenvalues of $B(t)$. Draw four plots of the eigenvalues of $B(t)$ when $t = 0, 1/3, 2/3, 1$. (Eigenvalues are continuous functions of the matrix entries.)

3. [15 pts] (a) Consider the following differential equation:

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \mathbf{u},$$

where $\mathbf{u}(0) = (1,1)^T$. Solve for $\mathbf{u}(t)$.

(b) The differential equation is modified to

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{u},$$

where $\mathbf{u}(0) = (1,1)^T$. Show that $\mathbf{u}$ now explodes as $t \to \infty$? Is there a set of initial conditions for $\mathbf{u}(0)$ so that $\mathbf{u}(t) \to 0$ as $t \to \infty$.

4. [10 pts] Let's play a game involving a frog. There are four lily pads. Start the frog at lily pad 1. At each lily pad the frog jumps forward by two lily pads with probability 2/3; otherwise, it moves directly to lily pad 2. If the frog overleaps lily pad 4, then the frog loses a point and returns to lily pad 1; otherwise, if the frog lands on lily pad 4, then it wins a point and returns to lily pad 1.

In the long term, does the frog win or lose? (Hint: Place an extra lily pad 5 for catching the overleaping frog. Find the stable equilibrum using a computer.)