pset 6

1 a) $\det(A) = 6$ and Cramer's rule gives

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\det\begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}}{\det(A)} = \frac{3}{6} = \frac{1}{2}.$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}}{6} = -\frac{2}{6} = -1/3$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{1}{6}$$

b) Without Cramer's rule we know that $\begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}$ is a solution.

$$x_1 = \frac{\det(B_1)}{\det(B)} = \frac{\det([\underline{b} | B_2 | B_3 | B_4])}{\det(B)}$$

since cols of $[b | B_2 | B_3 | B_4]$ are linear dependent we have $\det([b | B_2 | B_3 | B_4]) = 0$

~~$x_1 \Rightarrow \det$~~

Hence, $x_1 = 0$.

2.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 0 \\ 5 & 0 & 2 & 1 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 3 & -1 \end{bmatrix}}_{A} \begin{matrix} \\ ② \leftarrow ② - 5① \\ ③ \leftarrow ③ + ① \\ ④ \leftarrow ④ - 2① \end{matrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -10 & -13 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -3 & -3 & -1 \end{bmatrix} \begin{matrix} \\ \\ ③ \leftarrow ③ + \frac{3}{10}② \\ ④ \leftarrow ④ - \frac{3}{10}② \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -10 & -13 & 1 \\ 0 & 0 & -9/10 & 33/10 \\ 0 & 0 & 9/10 & -13/10 \end{bmatrix} \begin{matrix} \\ \\ \\ ④ \leftarrow ④ + ③ \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -10 & -13 & 1 \\ 0 & 0 & -9/10 & 33/10 \\ 0 & 0 & 0 & \cancel{20/10}\ 2 \end{bmatrix}$$

$$\det(A) = 1 \times -10 \times \frac{-9}{10} \times 2 = 18.$$

(b) Next matrix is same as part (a) but with two rows swapped

$$\det(B) = -18.$$

(c) Same as part (a), but row1 is now −row1.

$$\det(C) = -18.$$

3 (a) TRUE

(b) TRUE

~~(c) TRUE~~

(c) TRUE

(d) ~~TRUE~~ FALSE

Any valid reason is allowed.

For example:

(a) Property of det.

(b) AB is not invertible.

~~(c) They could do PA = LU for invertible matrices. OR A = QR for matrices with linear~~

(c) This is a hard question. Happy for a n=2 argument.

(d) if $A = {}_{m}\overset{n}{\boxed{}}$ then $AA^T = m \times m$ and by (b) is not invertible.

4 a)

$$(Q_2^T Q_1)^T (Q_2^T Q_1) = Q_1^T \underbrace{(Q_2 Q_2^T)}_{= I} Q_1$$

$$= Q_1^T Q_1 = I .$$

so $Q_2^T Q_1$ = orthogonal matrix

(b) $Q_1 R_1 = Q_2 R_2 \Rightarrow Q_2^T Q_1 = R_2 R_1^{-1}$

(because $Q_2^{-1} = Q_1^T$)

(c) Let D = upper-triangular and orthogonal

Let $D = [d_1 | \dots | d_n]$

$$d_1 = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } \|d_1\| = 1 \text{ so } d_1 = \begin{pmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} *_1 \\ *_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cancel{\text{and}},\ d_1^T d_2 = \pm *_1 = 0, \text{ so } *_1 = 0$$

$$\therefore d_2 = \begin{pmatrix} 0 \\ *_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \|d_2\| = 1, \text{ so } d_2 = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} *_1 \\ *_2 \\ *_3 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{aligned} d_1 d_3 &= 0 \Rightarrow *_1 = 0 \\ d_2^T d_3 &= 0 \Rightarrow *_2 = 0 \\ d_3^T d_3 &= 0 \Rightarrow *_3 = \pm 1 \end{aligned}$$

Keep going and we find

$$D = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$$

d) $B = Q_2^T Q_1 =$ orthogonal

$= R_2^{-1} R_1 =$ upper-triangular

By part (c): $B = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

(e) Given two QR's, $Q_1 R_1 = Q_2 R_2 = A$,

we showed that

$$Q_2^T Q_1 = B = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$$

I do <u>not</u> know the signs.

and $R_2^{-1} R_1 = B = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

$$\therefore \quad Q_1 = Q_2 \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$$

$$R_1 = R_2 \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$$

There are 2^n ways to pick signs ~~sign choices~~ in $B \Rightarrow 2^n$ QR's of A.

LOTS!