

Course 18.06: Problem Set 6

Due 4PM, Thursday 29th October 2015, in the boxes at E17-131.

This homework has 4 questions to hand-in. Write down all details of your solutions, not just the answers. Show your reasoning. Please staple the pages together and clearly PRINT your name, recitation section, and the name of your recitation instructor on the first page of the problem set. You know the collaboration rules by now.

This homework also has an online self-graded part.

Problems 1–4

1. [10 pts] (a) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$. Using Cramer's rule, find the solution to $A\underline{x} = (0, 0, 1)^T$.

Let B be a 4×4 matrix such that if B_i is the i th column of B , then $B_2 + 3B_3 - 2B_4 = \underline{b}$.
(b) Using Cramer's rule, show that if $\det(B) \neq 0$, then the first component of the solution to $B\underline{x} = \underline{b}$ is 0. (You also know this without Cramer's rule.)

2. [10 pts]

Using elimination or otherwise, calculate

$$\det \left(\begin{bmatrix} 1 & 2 & 3 & 0 \\ 5 & 0 & 2 & 1 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 3 & -1 \end{bmatrix} \right), \quad \det \left(\begin{bmatrix} 5 & 0 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 3 & -1 \end{bmatrix} \right), \quad \det \left(\begin{bmatrix} -1 & -2 & -3 & 0 \\ 5 & 0 & 2 & 1 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 3 & -1 \end{bmatrix} \right).$$

3. [10 pts]

Are the following statements true or false? Give a reason.

- (a) For $n \times n$ matrices A and B we have $\det(AB) = \det(A)\det(B)$.
(b) If A is a 4×5 matrix and B is a 5×4 matrix, then $\det(AB) = 0$.
(c) If u and v are $n \times 1$ vectors, then $\det(I + uv^T) = (1 + v^T u)$. Here, I is the $n \times n$ identity matrix.
(d) If the columns of A are linear independent, then $\det(AA^T) \neq 0$.

4. [15 pts] In this question we will investigate how many Q 's and R 's are there so that $A = QR$.

Let A be an $n \times n$ square and invertible matrix and suppose that $A = Q_1 R_1 = Q_2 R_2$, where Q_1 and Q_2 are orthogonal matrices and R_1 and R_2 are upper-triangular matrices.

- (a) Show that $Q_2^T Q_1$ is an orthogonal matrix.
(b) Show that $Q_2^T Q_1 = R_2 R_1^{-1}$.
(c) If D is an upper-triangular matrix and an orthogonal matrix, show that D must be a diagonal matrix where each diagonal entry is $+1$ or -1 .
(d) By setting $B = Q_2^T Q_1 = R_2 R_1^{-1}$, what form does B take? (You may use the fact that $R_2 R_1^{-1}$ is an upper-triangular matrix without justification.)
(e) How many QR factorizations does A have?