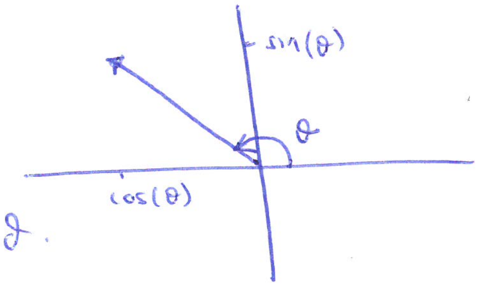
pset 5

1. Let's see:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$



rotates anticlockwise by θ.

(b) A is orthogonal as

$$\underbrace{A^{-1}}_{\text{rotate clockwise by } \theta} = \begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix} = A^T$$

(c) A^T rotates vectors clockwise by θ about origin.

(d) $$B^{-1} = \begin{bmatrix}-\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = B^T \; (= B)$$

$$B^{-1} = B^T \Rightarrow B \text{ is orthogonal.}$$

(e) let $Q = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ be orthogonal

$$Q^{-1} = Q^T$$

$$\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$$

$$(Q^{-1})^{-1} = Q$$

$$\frac{1}{(ad-bc)^2}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \text{so} \quad (ad-bc) = \pm 1$$

$$\therefore \pm \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\therefore a = \pm d, \quad c = \mp b, \quad ad - bc = \pm 1.$$

Hence, we must have $a^2 + b^2 = 1$,

setting $a = \pm \cos(\theta)$ solves the result.

2a) The linear system is

$$\underset{\underline{A}}{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} c \\ d \end{bmatrix} = \underset{\underline{b}}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^T \underline{b} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\therefore c = 4/3, \quad d = -1/2$$

(b) $\cancel{\|\underline{b} - A\begin{bmatrix} c \\ d \end{bmatrix}\|^2 =}$ $\quad \|\underline{b} - A\begin{bmatrix} c \\ d \end{bmatrix}\|^2 = 1/6$

So $1/\sqrt{6}$ is the least squares error

c) error is 0 as quadratic fits 3 points exactly.

3a) Let p be the projection. The plane $x-y-2z=0$ is the column space of $A=\begin{pmatrix}1&0\\1&2\\0&-1\end{pmatrix}$

Therefore,
$$p = A(A^TA)^{-1}A^T\underline{b} = \begin{pmatrix}13/2\\5/2\\2\end{pmatrix}$$

b)
$$P = A(A^TA)^{-1}A^T = \begin{pmatrix}5/6 & 1/6 & 1/3\\ 1/6 & 5/6 & -1/3\\ 1/3 & -1/3 & 1/3\end{pmatrix}$$

Check that $P^2 = P$.

c) $I-P$ projects on to the left nullspace of A, which spans $\begin{bmatrix}-1\\1\\2\end{bmatrix}$

d) $\underline{w} = \begin{pmatrix}1\\-5\\3\end{pmatrix}$ lies in the plane so $Pw = w$..

4 a) Let $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Then,

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2^T v_1}{v_1^T v_1} v_1 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3^T v_1}{v_1^T v_1} v_1 - \frac{x_3^T v_2}{v_2^T v_2} v_2$$

$$= \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{-6}{81} \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

Now normalize the vectors

$$v_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{81}} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$v_3 = \frac{v_3}{\|v_3\|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

(b) If A is $m \times n$ and $\dim(C(A)) < n$, then the columns of A are linearly dependent.

When we project off previous columns & we are going to end up with a zero vector, which is impossible to make of ~~norm~~ length 1 (without dividing by 0).

(c) Yes, the columns are linearly independent.

As we project off previous columns (and do linear combinations) we can never end up with ~~the~~ a zero column.