Course 18.06: Problem Set 5

Due 4PM, Thursday 22nd October 2015, in the boxes at E17-131.

This homework has 4 questions to hand-in. Write down all details of your solutions, not just the answers. Show your reasoning. Please staple the pages together and clearly PRINT your name, recitation section, and the name of your recitation instructor on the first page of the problem set.

Cooperation on problems is permitted, but all solutions must be written up independently and you must list your collaborators on the problem set. You should first try to solve each problem yourself, otherwise you will not learn much from hearing the solution.

This homework also has an online self-graded part.

Problems 1–4

1. [10 pts] For some $\theta \in [0, 2\pi]$, consider the following matrix:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

If v is a 2×1 vector, then Av is a vector formed by rotating v by an angle of θ about the origin. (For example, take $\theta = \pi/4$ and draw $v = (1, 0)^T$ and Av.)
(a) Does A rotate vectors clockwise or anti-clockwise?
(b) Is A orthogonal?
(c) Does A^T rotate vectors clockwise or anti-clockwise?
(d) Show that the following matrix is orthogonal and give an interpretation of what it does to vectors:

$$B = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(e) Show that for every 2×2 orthogonal matrix Q, there is a θ so that either $Q = A$ or $Q = B$.

2. [10 pts]
(a) Calculate (by hand) the least squares fit of the data $(-1, 2)$, $(0, 1)$, and $(1, 1)$ by the line $y = c + dx$.
(b) What is the least squares error from (a)?
(c) What is the least squares error if we fit $y = c + dx + ex^2$ to $(-1, 2)$, $(0, 1)$, and $(1, 1)$?

3. [15 pts]
(a) Find the projection of the vector $v = (6, 3, 3)^T$ onto the plane $x - y - 2z = 0$.
(b) Find the projection matrix P onto the plane $x - y - 2z = 0$, and verify that $P^2 = P$.
(c) If I is the identity matrix, then $I - P$ is also a projection matrix. Give the space that $I - P$ projects onto.
(d) Explain why $Pw = w$, where $w = (1, -5, 3)^T$ and P is the projection matrix in (b).

4. [15 pts]
(a) Apply Gram–Schmidt to the following vectors in $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Explain why the Gram–Schmidt process always fails (tries to divide by 0) on an $m \times n$ matrix A if $\dim(C(A)) < n$.
(c) Does the Gram–Schmidt process always succeed (never divides by 0) if $\dim(C(A)) = n$?