Problem set ~~#~~ 4

1. FALSE,

Consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad rref(B) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so $$rref(AB) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq rref(A)\,rref(B) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

2. I will do this the slow way so that the solution is clearer.

① Calculate rref(A).

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} ② \leftarrow ② - ① \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} ③ \leftarrow ③ + ②$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} ② \leftarrow -②$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} ① \leftarrow ① - 2①$$

$$\longrightarrow \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} = rref(A).$$

Therefore $$C(A) = \text{span of } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$ (go back to original matrix!)

$$N(A) = \text{span of } \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

free-variable

② Calculate rref(A^T)

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $$C(A^T) = \text{span of } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$N(A^T) = \text{span of } \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

① Calculate rref(B).

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & 2 & -2 & 3 \\ -2 & 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

Therefore, $C(A) = \text{span of } \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix} \right\} = \mathbb{R}^3$.

$N(A) = \text{span of } \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

free-variable

② Calculate rref (B^T)

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ -2 & -2 & 3 \\ 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$C(B^T) = \text{span of } \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}$.

$N(B^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3.

① $\sum_{i=1}^{6} A_{ii} = 0$ (= counts number of loops, nodes connected to themselves)

② $\sum_{i=1}^{6} (A^2)_{ii} = 2 \times$ # edges in the graph

③ $\sum_{i=1}^{6} (A^3)_{ii} = 6 \times$ # triangles in the graph.

For ②: To see this write out what $(A^2)_{ii}$ is as a sum:

$$(A^2)_{ii} = A_{i1}A_{1i} + A_{i2}A_{2i} + \dots + A_{i6}A_{6i}$$

$$= A_{i1}^2 + A_{i2}^2 + \dots + A_{i6}^2 \qquad (A \text{ is symmetric})$$

$= $ # edges coming out of i

$\sum_{i=1}^{6} (A^2)_{ii} = $ # edges coming out of node 1, 2, ..., 6

$= \underset{\uparrow}{2} \times$ # edges in graph.

an edge comes out of <u>2</u> nodes so an edge is double counted.

4.

$$C_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$ This is invertible

$$C_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$ Not invertible. cols 1 and 3 are the same.

$$C_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$ This is invertible.

$$C_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$ This is invertible.

b) If n is a multiple of 4, then

$$C_n \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \underline{0} \quad \text{Therefore, } C_n \text{ is } \underline{\text{not}} \text{ invertible.}$$

5 a)

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

b i)

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

sums up the columns of A.

(ii) LOOPS CORRESPOND TO LEFT NULL SPACE :

$$A^T \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A^T \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A^T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

(iii) $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is another loop in the circuit, also $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}$