pset 2 solutions

1 a)

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \text{swap} \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] ② \leftarrow -②$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

b)

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] ② \leftarrow ② - 3① \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] ② \leftarrow -\tfrac{1}{2}②$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] ① \leftarrow ① - 2②$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\therefore \quad B^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

c)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} ② \leftarrow ② - 4① \\ ③ \leftarrow ③ - 7① \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]$$

$$③ \leftarrow ③ + 2②$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & -15 & 2 & 1 \end{array}\right]$$

Complete failure!

C^{-1} does not exist!

b)

$$\left[\begin{array}{ccc|ccc} a & 1 & 1 & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array}\right] (3) \leftarrow \tfrac{1}{a}(3) \rightarrow \left[\begin{array}{ccc|ccc} a & 1 & 1 & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/a \end{array}\right] (2) \leftarrow (2) - (3)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} a & 1 & 1 & 1 & 0 & 0 \\ 0 & a & 0 & 0 & 1 & -1/a \\ 0 & 0 & 1 & 0 & 0 & 1/a \end{array}\right] (2) \leftarrow \tfrac{1}{a}(2)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} a & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/a & -1/a^2 \\ 0 & 0 & 1 & 0 & 0 & 1/a \end{array}\right] (1) \leftarrow (1) - (2) - (3)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & -1/a & 1/a^2 - 1/a \\ 0 & 1 & 0 & 0 & 1/a & -1/a^2 \\ 0 & 0 & 1 & 0 & 0 & 1/a \end{array}\right] (1) \leftarrow \tfrac{1}{a}(1)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & -\frac{1}{a^2} & \frac{1}{a^3} - \frac{1}{a^2} \\ 0 & 1 & 0 & 0 & 1/a & -1/a^2 \\ 0 & 0 & 1 & 0 & 0 & 1/a \end{array}\right]$$

so $D^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{1}{a^2} & \frac{1}{a^3} - \frac{1}{a^2} \\ 0 & 1/a & -1/a^2 \\ 0 & 0 & 1/a \end{bmatrix}$, assuming I had no complete failures.

pivots were a, a, a $\Rightarrow$ I'm not allow $a = 0$.

When $a = 0$, ~~D~~ my formula for D^{-1} gives $\frac{1}{0} = \infty$!!

2. I need

$$10x + 20y + 10z = 240$$
$$30x + 70y + 20z = 670$$
$$50x + 100y + 60z = 1280$$

$x =$ tap water, $y =$ bottle water, $z =$ industrial water

$$\left[\begin{array}{ccc|c} 10 & 20 & 10 & 240 \\ 30 & 70 & 20 & 670 \\ 50 & 100 & 60 & 1280 \end{array}\right] \begin{array}{l} ② \leftarrow ② - 3① \\ ③ \leftarrow ③ - 5① \end{array} \rightarrow \left[\begin{array}{ccc|c} 10 & 20 & 10 & 240 \\ 0 & 10 & -10 & -50 \\ 0 & 0 & 10 & 80 \end{array}\right]$$

By back substitution,

$\boxed{z = 8}$, $10y - 80 = -50$,

so $\boxed{y = 3}$

and $10x + 60 + 80 = 240$

$\boxed{x = 10}$

3 a) True

b) True

c) True

d) False

e) ~~True~~ False

f) True.

Any convincing reasons for (a), (b), (c), (f) are fine.

Any counterexample for (d) and (e) is fine too.

4. $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

b) AP swaps last two columns

PAP swaps last two rows and then last two columns,

P^2A ~~left~~ does not do anything $P^2 = I$.

c)

$$\begin{bmatrix} 3 & 4 & 1 \\ 3 & 4 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} ② \leftarrow ② - ① \\ ③ \leftarrow ③ - \frac{1}{3}① \end{matrix} \rightarrow \begin{bmatrix} 3 & 4 & 1 \\ 0 & 0 & 1 \\ 0 & -4/3 & 2/3 \end{bmatrix} \text{swap}$$

$$\rightarrow \begin{bmatrix} 3 & 4 & 1 \\ 0 & -4/3 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $U = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -4/3 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $P^2 = I$ so

$$E_{32} P E_{32} E_{21} = E_{32} P E_{32} (PP) E_{21} (PP)$$
$$= E_{32} (P E_{32} P)(P E_{21} P) P$$

(e) We can ~~reorder~~ swap the last two rows of A, at the start. Then we just do $(PE_{32}P)$ and $(PE_{21}P)$ and E_{32}.

(f). It does not know the right order. It can only work out the right order by actually doing the elimination.

(g) We know that

$$E_{32}(PE_{32}P)(PE_{31}P)PA = U$$

so
$$PA = (PE_{31}P)^{-1}(PE_{32}P)^{-1}E_{32}^{-1}U$$

$$L = (PE_{31}P)^{-1}(PE_{32}P)^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$