SOLUTIONS

18.06 Final Exam   Lecturer: Townsend   15th December, 2015

Your PRINTED name is: ____________________

Please CIRCLE your section:

R01	T	9	E17-128	Miriam Farber
R02	T	10	38-166	Sam Raskin
R03	T	10	E17-128	Miriam Farber
R04	T	11	38-166	Sam Raskin
R05	T	12	E17-133	Nate Harman
R06	T	1	E17-139	Tanya Khovanova
R07	T	2	E17-133	Tanya Khovanova
R08	T	2	38-166	Zach Abel
R09	T	3	38-166	Zach Abel

Grading
1:
2:
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Good luck! Hope you enjoyed 18.06.

Alex

1. (15 points in total. Each part is worth 3 points.)

Are the following statements below TRUE or FALSE? Give a brief reason.

(a) If all the entries of a square matrix A are positive, then A^{-1} exists.

FALSE

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(b) If Q is an orthogonal matrix, then $\det(Q) = \pm 1$.

TRUE

$1 = \det(I) = \det(Q^2) = \det(Q)^2 \quad \therefore \det(Q) = \pm 1.$

(c) If $A = U\Sigma V^T$ is the SVD of a square matrix A, then $A + I = U(\Sigma + I)V^T$ is necessarily the SVD of $A + I$. Here I is the identity matrix.

FALSE

$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ has singular values $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but $A+I$ has a zero singular value not $2, 2$.

(d) Let A be a real square matrix. If $\underline{x}$ is in $N(A)$ and $\underline{y}$ is in $C(A^T)$, then $\underline{x}^T\underline{y} = 0$.

TRUE. If $y \in C(A^T)$ then $\underline{y} = A^T\underline{z}$

so $\underline{x}^T\underline{y} = \underline{x}^T A^T \underline{z} = (A\underline{x})^T\underline{z} = \underline{0}^T\underline{z} = 0$

(e) If A and B are diagonalizable matrices, then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is diagonalizable.

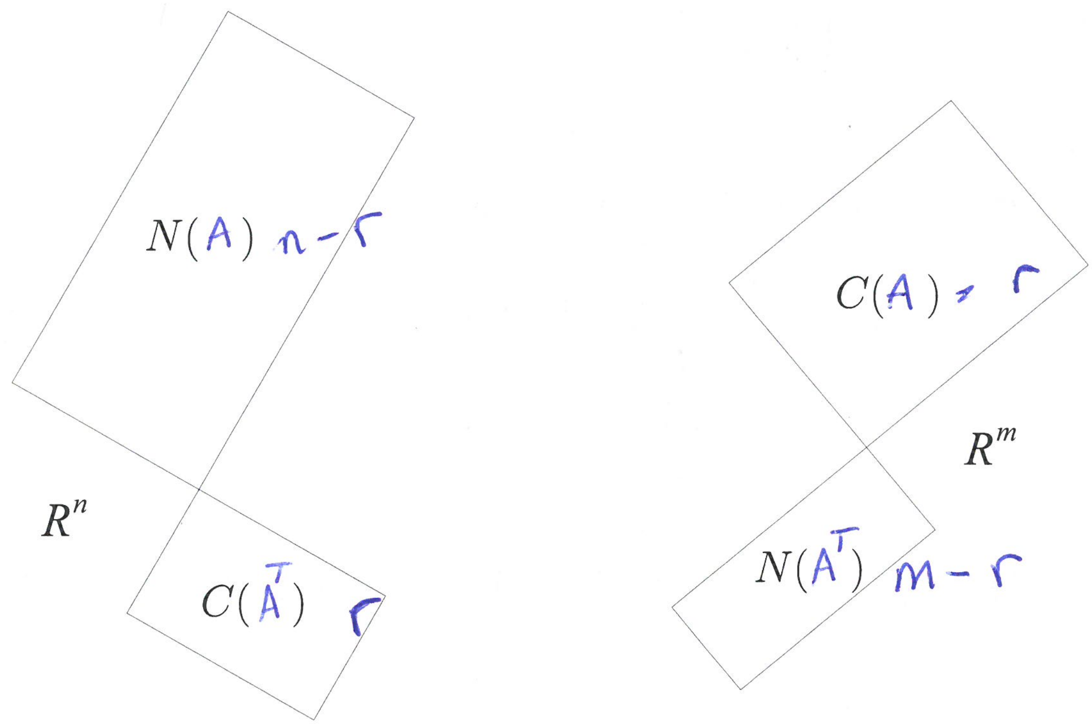
TRUE. Let $A = S_A \Lambda_A S_A^{-1}$ and $B = S_B \Lambda_B S_B^{-1}$.

Then
$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} S_A & 0 \\ 0 & S_B \end{bmatrix} \begin{bmatrix} \Lambda_A & 0 \\ 0 & \Lambda_B \end{bmatrix} \begin{bmatrix} S_A^{-1} & \\ & S_B^{-1} \end{bmatrix}$$

2. (10 points in total. Each part is worth 5 points.)

Let A be an $m \times n$ matrix with $m \geq n$ and $\text{rank}(A) = r < n$.

(a) On the figure below finish off labelling the four subspaces $C(A)$, $C(A^T)$, $N(A)$, and $N(A^T)$. Also give the dimension of each vector space.



(b) Calculate a singular value decomposition (SVD) of

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}.$$

$$A^TA = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} +4 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore \; \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$AV = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} U$$

$$\text{So } U = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

$$\therefore \; A = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. (15 points in total. (a) is worth 8, (b) is worth 3, and (c) is worth 4 points.)

(a) Calculate the reduced row echelon form of A and then give bases for $C(A)$ and $N(A)$, where A is the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 6 & 6 & 3 & 2 \\ 0 & -3 & 0 & -1 \end{bmatrix}.$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ C(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} \right\}$$

$$N(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

(b) Construct a 3×3 matrix such that the null space is spanned by $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ and the column space is spanned by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -5 \\ 1 & 1 & -3 \end{bmatrix}$$

fixes $C(A)$

so that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in N(A)$

(c) How many solutions (0, 1, or ∞) does $A\underline{x} = \underline{b}$ have? Please put your answers into the table below.

	$N(A) = \{\underline{0}\}$	$N(A) \neq \{\underline{0}\}$
$\underline{b}$ is in $C(A)$	1	∞
$\underline{b}$ is not in $C(A)$	0	0

4. (5 points total)

If A is a square matrix, then write down **five** conditions equivalent to the invertibility of A. Here are two (that do not count towards your five):

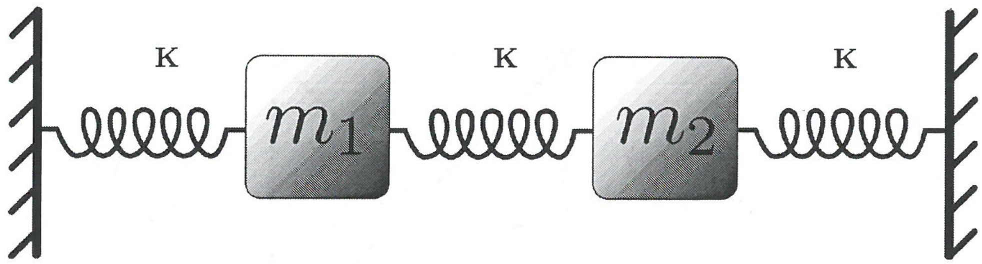
- There exists a matrix B such that $AB = BA = I$
- $\det(A) \neq 0$

Many possible answers here:

- No zero eigenvalue
- No zero singularvalue
- All ~~non~~ pivots are non-zero
- $A^T A$ is +ve def.
- $N(A) = \{0\}$
- $C(A) = \mathbb{R}^n$, $n = \#$ cols of A.

5. (10 points total. Each part is worth 5 points)

(a) Here is a diagram of two coupled springs:



which are governed by the differential equations

$$m_1 \frac{d^2u_1}{dt^2} = -2Ku_1(t) + Ku_2(t)$$
$$m_2 \frac{d^2u_2}{dt^2} = Ku_1(t) - 2Ku_2(t).$$

By setting $v_1 = \frac{du_1}{dt}$ and $v_2 = \frac{du_2}{dt}$, fill in the following 4×4 matrix:

$$\begin{bmatrix} \frac{du_1}{dt} \\ \frac{dv_1}{dt} \\ \frac{du_2}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2K}{m_1} & 0 & K/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/m_2 & 0 & -\frac{2K}{m_2} & 0 \end{bmatrix}}_{=A} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}.$$

(b) You are told that the two masses in (a) oscillate with a constant amplitude forever, what do you know about the eigenvalues of the 4×4 matrix A? Give a reason.

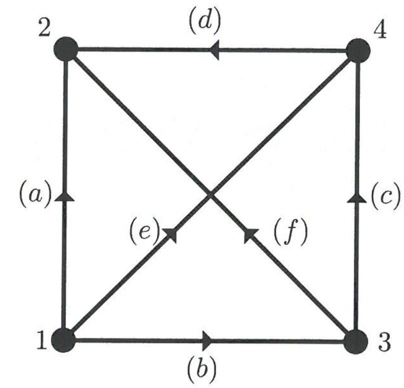
All four eigenvalues are purely ~~complex~~ imaginary. (well, at least the eigenvalues corresponding to excited eigenvectors).

This is because solution is

$$\underline{u}(t) = e^{At}\underline{u}(0) = c_1 e^{\lambda_1 t} v_1 + \dots + c_4 e^{\lambda_4 t} v_4$$

6. (10 points total. (a) is worth 8 and (b) is worth 2 points)

(a) Consider the following directed graph:



Write down the associated incidence matrix A, and give a basis for $N(A^T)$.

$$A = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Independent Loops:

$$N(A^T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

(b) Find the determinant of the following incidence matrix:

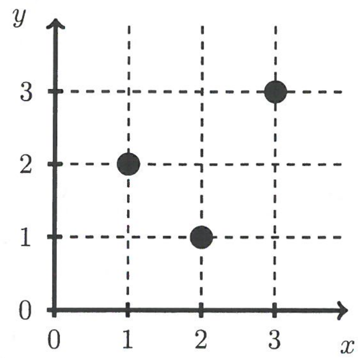
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

$$\det(A) = 0, \quad N(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

so A is not invertible.

7. (10 points total. Each part is worth 5 points)

(a) Find the least squares fit line $y = c + dx$ to the following three data points:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \qquad A^T \underline{b} = A^T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \end{bmatrix}$$

$\therefore$ Seek solution to $\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \end{bmatrix}$

By inspection (or elimination): $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

$$\boxed{y = c + \frac{1}{2} x}$$

(b) Let A be a matrix with linearly independent columns and consider the projection matrix $P = A(A^T A)^{-1} A^T$. What are the possible eigenvalues for P? Give reasons.

Possible eigenvalues are 0 and 1.

Since $P^2 = P$, $Pv = \lambda v \Rightarrow P^2 v = \lambda v = \lambda^2 v$.

$\lambda^2 = \lambda \Rightarrow \underline{\underline{\lambda = 0, 1}}$.

8. (10 points total. (a) is worth 7 and (b) is worth 3 points)

(a) Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$, where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & -3 \\ 0 & 3 & 2 \end{bmatrix} \text{②} \leftarrow \text{②} - 2\text{①} \rightarrow \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 3 & 2 \end{bmatrix} \text{③} \leftarrow \text{③} + 3\text{②}$$

$$\rightarrow \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$\therefore \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) Let A be a symmetric positive definite matrix. Which one of the tests for a symmetric positive definite matrix directly guarantees that $A = LU$, i.e., elimination never has a temporary or complete failure? You must give a reason for credit.

The det test. Since ① det $1\times1 > 0$, 1^{st} pivot is nonzero.
② det $2\times2 > 0$, so 2^{nd} pivot cannot be zero either

and so on...

det $(n \times n) > 0$ so n^{th} pivot is nonzero.

9. (15 points total. (a) is worth 8 and (b) is worth 7 points)

(a) Find the eigenvalue decomposition of A and calculate A^{2015}, where

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}.$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & -3 \\ -1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda) - 3$$
$$= \lambda^2 - 6\lambda + 5$$
$$= (\lambda-1)(\lambda-5)$$

$\lambda_1 = 5, \lambda_2 = 1$

$$A-\lambda_1 I = \begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix} \quad \therefore \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A-\lambda_2 I = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \quad \therefore \quad v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$A^{2015} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5^{2015} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}^{-1}.$$

(b) Let T be the transformation

$$T : \{\text{ polynomials of degree} \leq 4 \} \quad \to \quad \{ \text{ polynomials of degree} \leq 4 \},$$

$$T(p) = (x-1)\frac{dp}{dx}.$$

Show that T is a linear transformation and write down a matrix representing T with basis $\{1, x, x^2, x^3, x^4\}$ for the input and output spaces.

$T(1) = 0$, $T(x) = (x-1)$, $T(x^2) = 2x^2 - 2x$

$T(x^3) = 3x^3 - 3x^2$, $T(x^4) = 4x^4 - 4x^3$

$$\therefore \quad A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 & +4 \end{bmatrix}$$

5×5.