SOLUTIONS

18.06 Exam III **Lecturer: Townsend** **2nd December, 2015**

Your PRINTED name is: ______________________________

Please CIRCLE your section:

R01	T	9	E17-128	Miriam Farber
R02	T	10	38-166	Sam Raskin
R03	T	10	E17-128	Miriam Farber
R04	T	11	38-166	Sam Raskin
R05	T	12	E17-133	Nate Harman
R06	T	1	E17-139	Tanya Khovanova
R07	T	2	E17-133	Tanya Khovanova
R08	T	2	38-166	Zach Abel
R09	T	3	38-166	Zach Abel

Grading 1:

2:

3:

4:

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1. (20 points in total. Each part is worth 5 points.)

Are the following statements below TRUE or FALSE? **Give a brief reason.**

(a) If A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

TRUE if $A\underline{x} = \lambda \underline{x}$ then $\underline{x} = \lambda A^{-1}\underline{x}$ and $A^{-1}\underline{x} = \frac{1}{\lambda}\underline{x}$.

(b) If A is a positive definite matrix, then $A^T + I$ is also a positive definite matrix.

TRUE if A is +ve def, then so is A^T,

$A^T + I$ ~~is eig~~ has eigenvalues of A^T shifted by +1.

(c) The following matrix is positive definite:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

FALSE. Fails det test

1x1 det = 2

2x2 det = 3

3x3 det = -2

(d) Let A be a real *skew-symmetric* $n \times n$ matrix, i.e., $A^T = -A$. If λ is an eigenvalue of A, then so is $-\lambda$.

TRUE

$$0 = \det(A - \lambda I) = \det((A-\lambda I)^T) = \det(A^T - \lambda I) = \det(-A - \lambda I)$$
$$= (-1)^n \det(A + \lambda I)$$

2. (20 points in total. Each part is worth 10 points.)

(a) Calculate e^{At}, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$\therefore \quad A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

(b) Using your answer from part (a), solve the system of differential equations:

$$\frac{du}{dt} = u(t) + v(t),$$

$$\frac{dv}{dt} = 0u(t) + v(t),$$

where $u(1) = 1$ and $v(1) = 0$.

Warning: You have been given conditions $u(1) = 1$ and $v(1) = 0$ that are **not** at $t = 0$. Adjust your solution accordingly.

$$\begin{bmatrix} u \\ v \end{bmatrix} = e^{A(t-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{A(t-1)} = \begin{bmatrix} e^{t-1} & (t-1)e^{t-1} \\ 0 & e^{t-1} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e^{t-1} \\ 0 \end{bmatrix}$$

Happy for them to spot this solution with part a)

3. (30 points total. Each part is worth 10 points)

(a) Consider the transformation

$$T: \quad \{\text{ polynomials of degree } \leq n \} \quad \to \quad \mathbb{R}$$

given by

$$T(p) = \int_0^1 p(s)ds.$$

Show that T is a linear transformation.

$T(cp) = \int_0^1 cp(s)ds = c\int_0^1 p(s)ds = cT(p)$

$T(p+q) = \int_0^1 (p+q)(s)ds = \int_0^1 p(s)ds + \int_0^1 q(s)ds$

$= T(p) + T(q)$

(b) For the linear transformation T from part (a), you are given the relation

$$T(x^k) = \int_0^1 x^k dx = \frac{1}{k+1}, \qquad k \geq 0.$$

Pick a basis for the input space, a basis for the output space, and find the corresponding matrix that represents T.

Input space basis : $\{1, x, \ldots, x^n\}$

Output space basis : $\{1\}$

$T(x^k) = \frac{1}{k+1}$

so $A = \left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}\right]$

(c) Let A be a 2×2 matrix such that

$$A \begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}, \quad A \begin{bmatrix} c \\ d \end{bmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix} + 2 \begin{bmatrix} c \\ d \end{bmatrix},$$

where a, b, c, and d are real numbers.

Are A and $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ similar? Give a condition on a, b, c, and d, if necessary.

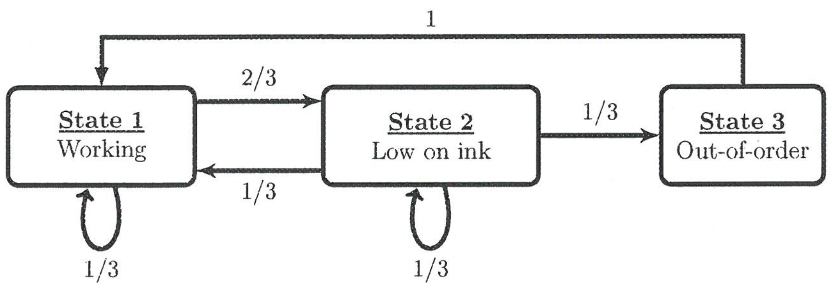
$$A \begin{bmatrix} c & a \\ d & b \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} c & a \\ d & a \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c & a \\ d & b \end{bmatrix}^{-1}$$

assuming $ad - bc \neq 0$

4. (30 points total. Each part is worth 10 points)

(a) Here is a flowchart for the status of the MIT exam printing machine:



The fractions on the arrows indicate the probability that the machine moves from one particular state to another.

(i) Define the term *Markov matrix.*
(ii) Why does a Markov matrix always have 1 as an eigenvalue?
(iii) Write down the Markov matrix associated to the flowchart above.

(i) A = $n \times n$ matrix is Markov if

1. ~~Column~~ Entries in each column sum to 1
2. Nonnegative entries.

(ii) $A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ so eigenvalue of A^T is 1

and A and A^T have the same eigenvalues.

(iii)

$$A = \begin{bmatrix} 1/3 & 1/3 & 1 \\ 2/3 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

(b) The printer is upgraded. The associated Markov matrix is now:

$$A = \begin{bmatrix} 1/2 & 3/4 & 1 \\ 0 & 1/4 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}.$$

In the long-run what proportion of time is the printer in each state assuming the printer starts off working?

We just need the eigenvector corresponding to eigenvalue 1:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore \quad \begin{aligned} -\tfrac{1}{2}x + \tfrac{3}{4}y + z &= 0 \\ -\tfrac{3}{4}y &= 0 \\ \tfrac{1}{2}x - z &= 0 \end{aligned}$$

$$\therefore \quad y = 0, \quad z = 1, \quad x = 2$$

So eigenvector is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

The printer spends twice as long in State 1 as State 3 in the long-run. It spends no time in State 2.

(c) For a standard color inkjet printer, the associated Markov matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and linear independent eigenvectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$, where $\lambda_1 = 1$, $\lambda_2 = -1$, and $|\lambda_3| < 1$. What is the long-run behavior of the printer?

$$\underline{x}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3$$

then $$A^k \underline{x}_0 = c_1 \lambda_1^k \underline{v}_1 + c_2 \lambda_2^k \underline{v}_2 + c_3 \lambda_3^k \underline{v}_3$$

For large k: $$A^k \underline{x}_0 \approx c_1 \underline{v}_1 + c_2 (-1)^k \underline{v}_2 .$$

"oscillatory state".