SOLUTIONS

18.06 Exam II **Lecturer: Townsend** **6th November, 2015**

Your PRINTED name is: ____________________

Please CIRCLE your section:

R01	T	9	E17-128	Miriam Farber
R02	T	10	38-166	Sam Raskin
R03	T	10	E17-128	Miriam Farber
R04	T	11	38-166	Sam Raskin
R05	T	12	E17-133	Nate Harman
R06	T	1	E17-139	Tanya Khovanova
R07	T	2	E17-133	Tanya Khovanova
R08	T	2	38-166	Zach Abel
R09	T	3	38-166	Zach Abel

Grading 1:

2:

3:

4:

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1. (20 points in total. Each part is worth 5 points.)

Are the following statements below TRUE or FALSE? **Give a brief reason.**

(a) If A is a square matrix and $\det(A) = \pm 1$, then A is an orthogonal matrix.

FALSE

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$$

(b) If A is square and $A = QR$, then $|\det(A)|$ = product of diagonal entries of R. Here, Q is an orthogonal matrix and R is upper-triangular.

~~TRUE~~ FALSE

$$|\det(A)| = |\det(QR)| = |\det(Q)\det(R)| = |\det(R)|$$

So $|\det(A)| = |\text{prod of diag}(R)|$, Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) If Q is an $m \times n$ matrix with orthonormal columns and $m \geq n$, then $QQ^T = I_m$. Here, I_m is the $m \times m$ identity matrix.

FALSE: if $m > n$, then QQ^T is of rank n $\neq I_m$.

(d) If A is a matrix with independent columns and $P = A(A^TA)^{-1}A^T$, then $I - P$ projects onto the left null space $N(A^T)$.

TRUE: P projects onto $C(A)$.

So $I-P$ projects onto orthogonal complement $= N(A^T)$.

2. (20 points in total. Each part is worth 10 points.)

(a) Calculate the determinant of the following 4×4 matrix:

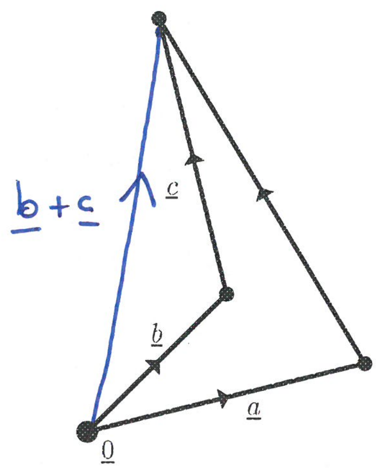
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

You must **show** your calculations to receive full credit.

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= -6 + 1 = -5.$$

(b) Here is a spaceship-shaped polygon in $\mathbb{R}^2$:



What is the area of the spaceship? (Hint: Write down a formula using determinants.)

$$\text{Area} = \frac{1}{2}\left|\det\left(\left[\underline{a} \mid \underline{b}+\underline{c}\right]\right)\right| - \frac{1}{2}\left|\det\left(\left[\underline{b} \mid \underline{b}+\underline{c}\right]\right)\right|$$

3. (30 points total. Each part is worth 10 points)

(a) Let $\underline{q}_1, \ldots, \underline{q}_n$ be a set of nonzero orthogonal vectors in $\mathbb{R}^m$. Show that $\underline{q}_1, \ldots, \underline{q}_n$ are linearly independent.

Suppose $c_1 \underline{q}_1 + \cdots + c_n \underline{q}_n = 0$

need to show $c_1 = 0, \ldots, c_n = 0$.

$\cancel{\underline{q}_1^T (c_1 \underline{q}_1 + \cdots + c_n)}$

Let $Q = [\underline{q}_1 | \cdots | \underline{q}_n]$, $Q \underline{c} = \underline{0}$

so $Q^T Q \underline{c} = \underline{c} = \underline{0}$.

$\Rightarrow$ $q_1, \ldots, q_n$ linearly independent.

(b) Find Q and R in a $A = QR$ factorization of A, where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \underline{v}_2 = \underline{a}_2 - \frac{\underline{a}_2^T \underline{v}_1}{\underline{v}_1^T \underline{v}_1} \underline{v}_1$

$= \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$\underline{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{q}_2 = \frac{1}{\sqrt{8}} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

$\therefore \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -2/\sqrt{8} \\ 1/\sqrt{2} & 2/\sqrt{8} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{8} \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}$

(c) Calculate a and b in the least squares best fit equation

$$y = a + b\cos\left(\tfrac{\pi}{2}x\right)$$

to the data $(x, y) = (-1, 0)$, $(0, 1)$, and $(2, 3)$.

(Note that $\cos(\frac{\pi}{2}) = 0$, $\cos(-\frac{\pi}{2}) = 0$, $\cos(0) = 1$, and $\cos(\pi) = -1$.)

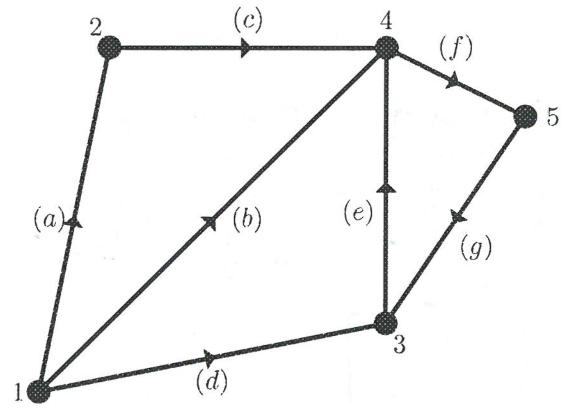
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A^T\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\therefore \quad A^TA\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \therefore \boxed{\begin{matrix} a = 4/3 \\ b = -1 \end{matrix}}.$$

4. (30 points total. Each part is worth 15 points)

(a) Here is an electrical circuit where each edge has a capacity/resistance of 1.



(i) Write down the incidence matrix of the graph. (Please use the same ordering of the nodes 1-5 and edges (a)-(f).)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

(ii) If $\underline{x} = (x_1, x_2, x_3, x_4, x_5)^T$ is a vector of potentials at the nodes, then what are the physical interpretations of the vectors $\underline{e} = A\underline{x}$ and $\underline{w} = A^T\underline{e}$?

$\underline{e} = A\underline{x}$ ~~is~~ ~~of~~ potential differences.

$\underline{w} = A^T\underline{e}$ current flowing out of nodes

(b) Here is the incidence matrix for a graph:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Give a **basis** for the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, and $N(A^T)$. (Hint: There is no need to compute the reduced-row echelon form because A is an incidence matrix.)

$C(A)$: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} +1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

$N(A)$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$N(A^T)$: $\boxed{\text{loops}}$ $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$C(A^T)$: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

GRAPH :