18.06 Exam I **Lecturer: Townsend** **5th October, 2015**

Your PRINTED name is: ________________________

Please CIRCLE your section:

R01	T	9	E17-128	Miriam Farber
R02	T	10	38-166	Sam Raskin
R03	T	10	E17-128	Miriam Farber
R04	T	11	38-166	Sam Raskin
R05	T	12	E17-133	Nate Harman
R06	T	1	E17-139	Tanya Khovanova
R07	T	2	E17-133	Tanya Khovanova
R08	T	2	38-166	Zach Abel
R09	T	3	38-166	Zach Abel

Grading 1:

2:

3:

4:

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1. (20 points in total. Each part is worth 5 points.)

Are the following statements below TRUE or FALSE? Give a brief reason.

(a) If A is an $m \times n$ matrix and $m \neq n$, then $(A^T)^T = A$. Here, A^T is the *transpose* of A.

TRUE. A^T flips across diagonal (reflection)

$(A^T)^T$ flips twice $\Rightarrow (A^T)^T = A$.

(b) Let A and B be square matrices that are not symmetric. If AB is a symmetric matrix, then $AB = B^T A^T$.

TRUE. $(AB)^T = AB$ as AB is symmetric

$\therefore AB = (AB)^T = B^T A^T$.

(c) If A^{-1} exists, then A^T is an invertible matrix.

TRUE. $A^{-1}A = I \Rightarrow I^T = (A^{-1}A)^T = A^T(A^{-1})^T$

$\therefore (A^{-1})^T = (A^T)^{-1}$.

(d) If A^{-1} exists, then elimination on A will proceed to completion without permuting any rows of A.

FALSE. Elimination may need row pivoting from temporary failures. For example, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

2. (20 points in total. Each part is worth 10 points.)

(a) Using the method of Gauss–Jordan and showing your work, find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Please verify your answer by checking that $A^{-1}A = I$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \textcircled{2} \leftarrow \textcircled{2} - 2\textcircled{1} \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] \textcircled{2} \leftarrow -\tfrac{1}{3}\textcircled{2} \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right] \textcircled{1} \leftarrow \textcircled{1} - 2\textcircled{2}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}.$$

$$A^{-1}A = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

(b) Without doing Gauss–Jordan, write down the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

"Inverse of block diagonal matrix is block matrix with inverse blocks"

$$A^{-1} = \begin{bmatrix} -1/3 & 2/3 & & & & \\ 2/3 & -1/3 & & & & \\ & & -1/3 & 2/3 & & \\ & & 2/3 & -1/3 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix},$$

3. (30 points total. Each part is worth 10 points)

(a) Calculate the reduced row echelon form of A, where

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 6 & 3 \end{bmatrix}.$$

Using your answer, describe the column space and nullspace of A.

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 6 & 3 \end{bmatrix} \begin{matrix} \textcircled{2} \leftarrow \textcircled{2} - \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} - \textcircled{1} \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \textcircled{3} \leftarrow \textcircled{3} - 2\textcircled{2}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \textcircled{2} \leftarrow \textcircled{2}/2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \textcircled{1} \leftarrow \textcircled{1} - 2\textcircled{2}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$N(A) = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Write down ALL the solutions to

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\underline{\text{All sol}^{\text{ns}}} = \left(\begin{matrix}\text{particular} \\ \text{sol}\end{matrix}\right) + y \; 4, \quad y \in N(A)$$

$$= \begin{bmatrix} 0 \\ 9/2 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) If A is an $n \times n$ matrix and A^{-1} exists, then what is the column space and null space of A? Write down a basis for $C(A)$.

$C(A) = \mathbb{R}^n$

$N(A) = \{[\underline{0}]\}$

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}.$

4. (30 points total. Each part is worth 10 points)

(a) Solve the following linear system for x, y, and z:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 3 \\ 0 & 3 & -1 & 4 \end{array}\right] \text{②} \leftarrow \text{②} - 2\text{①} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 3 & -1 & 4 \end{array}\right] \text{swap}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -1 & 4 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$\therefore$ $\underline{z = +1}$, $3y - 1 = 4 \Rightarrow \underline{y = 5/3}$

$x + \frac{5}{3} + 1 = 2 \Rightarrow$ $\underline{x = -2/3}$

(b) Using row manipulations, calculate an $A = UL$ decomposition for the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}.$$

Note that U and L in $A = UL$ are reversed. Here, U is an upper-triangular matrix and L is a lower-triangular matrix.

I want to put a zero in $A(1,2)$ position.

$$\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} \text{①} \leftarrow \text{①} - 4\text{②} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

So $$\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{\text{upper}} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{lower.}}$$

(c) Given 1000 vectors $\underline{b}_1, \ldots, \underline{b}_{1000}$, describe a way to solve ALL the 1000 linear systems,

$$A\underline{x}_1 = \underline{b}_1, \quad A\underline{x}_2 = \underline{b}_2, \quad \ldots, \quad A\underline{x}_{1000} = \underline{b}_{1000},$$

without doing elimination on A more than once. (The matrix A is invertible.)

Many ways:

① Do elimination on $[A \mid I]$ to calculate A^{-1}.

Then $\underline{x}_K = A^{-1}\underline{b}_K$

② ① Calculate $PA = LU$

③ Solve $PA\underline{x}_k = LU\underline{x}_k = \underline{b}_K$ by substitution (twice)

③ Do elimination to $[A \mid \underline{b}_1 | \underline{b}_2 | \cdots | \underline{b}_{1000}]$