Problem Set 9 Solutions

Question 1: Section 6.4, Problem 4, page 338

Let $Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} -5 & 0 \\ 0 & 10 \end{pmatrix}$. Then Q is orthogonal and $A = Q\Lambda Q^T$.

One could also swap the columns of Q and their sign. If one swaps the columns, one must swap -5 and 10.

Question 2: Section 6.4, Problem 6, page 338.

$\frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$,
$\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$.

Question 3: Section 6.4, Problem 9, page 338.

Let A be a real 3×3 matrix. The characteristic polynomial $\det(A - \lambda I)$ is a real cubic polynomial in λ and by the fundamental theorem of algebra there exists a root $\lambda_1 \in \mathbb{C}$. If $\lambda_1 \in \mathbb{R}$ we are done since this gives a real eigenvalue of A. If $\lambda_1 \notin \mathbb{R}$ then $\lambda_2 = \overline{\lambda_1}$ is a root to the polynomial, distinct from λ_1. Now, $(\lambda - \lambda_1)(\lambda - \lambda_2)$ is a real quadratic dividing $\det(A - \lambda I)$. By performing the division we find a *real* linear factor $(\lambda - \lambda_3)$ of $\det(A - \lambda I)$. This gives a real eigenvalue λ_3 of A.

Question 4: Section 6.4, Problem 18, page 339.

1.

Lemma. *Suppose that A is a real symmetric matrix. Then the column space of A is orthogonal to the nullspace of A.*

Proof. Suppose that A is a real symmetric matrix, that x is in the column space and that y is in the nullspace. Then there exists z with $Az = x$, $Ay = 0$ and we have

$$x \cdot y = (Az) \cdot y = (Az)^T y = (z^T A^T) y = z^T (Ay) = z^T (0) = 0.$$

□

2.

Lemma. *Suppose that A is a square matrix and that x is an eignvector for A with eigenvalue λ. Then x is an eigenvector for $A - \beta I$ with eigenvalue $\lambda - \beta$.*

Proof. Suppose that x is an eignvector for A with eigenvalue λ. By definition this means that $x \neq 0$ and $Ax = \lambda x$. Thus, $x \neq 0$ and $(A - \beta I)x = Ax - \beta x = \lambda x - \beta x = (\lambda - \beta)x$, so that x is an eigenvector for $A - \beta I$ with eigenvalue $\lambda - \beta$. □

Applying the first argument to $A - \beta I$:

Suppose that A is a real symmetric matrix, that x is an eigenvector for A with eigenvalue λ and that y is an eigenvector for A with eigenvalue β distinct from λ. Then the lemma tells us that x is an eigenvector for $A - \beta I$ with eigenvalue $\lambda - \beta$ (which is nonzero) and that y is an eigenvector for $A - \beta I$ with eigenvalue 0. By part 1 we see that x and y are orthogonal.

Question 5: Section 6.5, Problem 1, page 350.

(i) λ_1 and λ_2 have the same sign because their product $\lambda_1\lambda_2$ equals $ac - b^2 > 0$.

(ii) That sign is positive because $\lambda_1 + \lambda_2$ equals $a + c > 0$.

Question 6: Section 6.5, Problem 5, page 350

$f(x, y) = x^2 + 4xy + 3y^2 = (x + 2y)^2 - y^2$, so $f(-2, 1) = -1$.
This corresponds to the LDL^T decomposition

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We can also do this using the $Q\Lambda Q^T$ decomposition. The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 2+\sqrt{5}$ and $\lambda_2 = 2 - \sqrt{5}$ with corresponding eignevectors

$$v_1 = (1, (1 + \sqrt{5})/2)^T \text{ and } v_2 = (1, (1 - \sqrt{5})/2)^T,$$

respectively. Since $v_1 \cdot v_1 = (5 + \sqrt{5})/2$ and $v_2 \cdot v_2 = (5 - \sqrt{5})/2$ we obtain

$$f(x, y) = x^2 + 4xy + 3y^2 = \frac{2(2 + \sqrt{5})}{5 + \sqrt{5}} \left(x + \left(\frac{1 + \sqrt{5}}{2} \right) y \right)^2 + \frac{2(2 - \sqrt{5})}{5 - \sqrt{5}} \left(x + \left(\frac{1 - \sqrt{5}}{2} \right) y \right)^2.$$

Since $\frac{2(2+\sqrt{5})}{5+\sqrt{5}} > 0$ and $\frac{2(2-\sqrt{5})}{5-\sqrt{5}} < 0$ we can take the square roots of the absolute values of the coefficients into the parentheses to obtain a difference of squares.

$$f(-1 - \sqrt{5}, 2) = 20 \cdot \frac{2(2 - \sqrt{5})}{5 - \sqrt{5}} < 0.$$

Question 7: Section 6.5, Problem 6, page 350.

$f(x, y) = (x, y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are 1 and -1.

Question 8: Section 6.5, Problem 15, page 351.

You were only required to give one method but here are two.

1. Assume A and B are positive definite $n \times n$ matrices. Then $x^T Ax,\ x^T Bx > 0$ for all nonzero $x \in \mathbb{R}^n$. Thus $x^T (A + B)x = x^T Ax + x^T Bx > 0$ for all nonzero $x \in \mathbb{R}^n$, which means that $A + B$ is positive definite.

2. Assume A and B are positive definite $n \times n$ matrices. Then there exist matrices R and S with $A = R^T R$ and $B = S^T S$. R is $k \times n$ for some k and S is $m \times n$ for some m. $\begin{pmatrix} R \\ S \end{pmatrix}$ is $(k+m) \times n$. $\begin{pmatrix} R \\ S \end{pmatrix}^T = \begin{pmatrix} R^T & S^T \end{pmatrix}$ and so $\begin{pmatrix} R \\ S \end{pmatrix}^T \begin{pmatrix} R \\ S \end{pmatrix} = R^T R + S^T S = A + B$. This implies that $A + B$ is positive definite.

Question 9: Section 6.5, Problem 26, page 353.

(a) Let $A = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{pmatrix}$.

To row reduce A we subtract two lots of the second row from the third row and we find the pivots are $9, 1$ and 4. This gives us

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

So $A = C^T C$ where $C^T = L\sqrt{D}$. Thus

$$C = \sqrt{D} L^T = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{pmatrix}$.

We see that the relevant row reduction matrix to consider is $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ and the pivots are $1, 1$ and 5, so that $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. We have $A = C^T C$ where $C^T = L\sqrt{D}$. Thus

$$C = \sqrt{D} L^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{pmatrix}.$$

Question 10: Section 6.6, Problem 2, page 360.

Let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $M = M^{-1}$ and $M \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} M = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.