18.06 Problem Set 8. Solutions

Problem 1. Section 6.2, Problem 14, page 309.

The matrix $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is ______. Change one entry to make A diagonalizable. Which entries could you change?

Solution. The rank of $A - 3I$ is 1. A matrix is diagonalizable if for each eigenvalue λ the rank r of $A - \lambda I$ equals n (the size of the matrix) minus the multiplicity of λ. For example, a matrix with all distinct eigenvalues is diagonalizable. If a diagonal entry in A is changed, the new matrix is an upper triangular matrix with different diagonal entries, and has, therefore, distinct eigenvalues. If the entry a_{21} is changed to x, then the new eigenvalues are solutions to the equation $(3 - \lambda)^2 = x$ and are distinct. If the entry a_{12} is changed to a non-zero value, then the new matrix is not diagonalizable for the same reason why A is not diagonalizable. If the entry a_{12} is changed to zero then the new matrix is itself diagonal.

Problem 2. Section 6.3, Problem 3, page 325.

(a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $u' = Au$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$. Solve $\frac{d\boldsymbol{u}}{dt} = \left[\begin{smallmatrix} -2 & 3 \\ 2 & -3 \end{smallmatrix}\right] \boldsymbol{u}$ with $\boldsymbol{u}(0) = \left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]$. What is $\boldsymbol{u}(\infty)$?

Solution.

(a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) One of the eigenvalues of $A = \left[\begin{smallmatrix} -2 & 3 \\ 2 & -3 \end{smallmatrix}\right]$ is 0 (see the first part). The second one can be found by using the trace. It is equal to -5. The corresponding eigenvectors are $\boldsymbol{x}_1 = (3, 2)$ and $\boldsymbol{x}_2 = (1, -1)$. Then the usual 3 steps:

1. Write $\boldsymbol{u}(0)$ as a linear combination of the eigenvectors: $\boldsymbol{u}(0) = [4\ 1]^T = [3\ 2]^T + [1\ -1]^T = \boldsymbol{x}_1 + \boldsymbol{x}_2$.

2. Use the solutions to the equation generated by the eigenvectors. If $\boldsymbol{u}(0) = \boldsymbol{x}_i$, then $\boldsymbol{u}(t) = e^{\lambda_i t}\boldsymbol{x}_i$.

3. The solution $\boldsymbol{u}(t) = e^{0t}\boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2 = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ approaches the steady state $\boldsymbol{x}_1 = (3, 2)$ as $t \to \infty$.

Problem 3. Section 6.3, Problem 7, page 326. Suppose P is the projection matrix onto the 45° line $y = x$ in $\mathbb{R}^2$. What are its eigenvalues? If $d\boldsymbol{u}/dt = -P\boldsymbol{u}$ (notice minus sign) can you find the limit of $\boldsymbol{u}(t)$ at $t = \infty$ starting from $\boldsymbol{u}(O) = (3, 1)$?

Solution. A projection matrix has eigenvalues 1 and 0. Eigenvectors corresponding to eigenvalue 1 fill the subspace that P projects onto: here $\boldsymbol{x}_1 = (1, 1)$. Eigenvectors corresponding to eigvenvalue 0 fill the perpendicular subspace: here $\boldsymbol{x}_2 = (1, -1)$. Then the usual 3 steps:

1. Write $\boldsymbol{u}(0)$ as a linear combination of the eigenvectors: $\boldsymbol{u}(0) = [3\ 1]^T = 2[1\ 1]^T + [1\ -1]^T = 2\boldsymbol{x}_1 + \boldsymbol{x}_2$.

2. Use the solutions to the equation generated by the eigenvectors. Do not forget the minus sign: the eigenvalues of $-P$ are -1 and 0.

3. The solution $\boldsymbol{u}(t) = e^{-t}2\boldsymbol{x}_1 + e^{0t}\boldsymbol{x}_2 = e^{-t}[2\ 2]^T + e^{0t}[1\ -1]^T$ approaches the steady state $\boldsymbol{x}_2 = (1, -1)$ as $t \to \infty$.

Problem 4. Section 6.3, Problem 10, page 326. Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{u} = (y, y')$: $\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = A\boldsymbol{u}$. What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

Solution. $\frac{d}{dt} = \left[\begin{smallmatrix} y \\ y' \end{smallmatrix}\right] = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \left[\begin{smallmatrix} 0 & 1 \\ 4 & 5 \end{smallmatrix}\right] \left[\begin{smallmatrix} y \\ y' \end{smallmatrix}\right]$. Therefore, $A = \left[\begin{smallmatrix} 0 & 1 \\ 4 & 5 \end{smallmatrix}\right]$. The eigenvalues can be found by calculating the determinant of $A - \lambda I$, or by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4$. Naturally both methods produce the same equation $\lambda^2 = 5\lambda + 4$. The eigenvalues are $(5 \pm \sqrt{41})/2$.

Problem 5. Section 6.3, Problem 23, page 328. Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 21-22 and 19. (If $AB = BA$, all three are the same.)

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution. $e^A = \begin{bmatrix} e & 4(e-1) \\ 0 & 1 \end{bmatrix}$ from problem 21 and $e^B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ from problem 19. By direct multiplication

$$e^A e^B = \begin{bmatrix} e & -4 \\ 0 & 1 \end{bmatrix} \neq e^B e^A = \begin{bmatrix} e & 4e-8 \\ 0 & 1 \end{bmatrix} \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}.$$