18.06 Problem Set 7
Solutions

Problem 1. Section 5.2, Problem 3, page 263. Clarification. In this problem x's symbolize 5 nonzero entries (not necessarily equal).

Each cofactor C_{ij} is given by: $C_{ij} = (-1)^{i+j} det(M_{ij})$, where the submatrix M_{ij} is obtained from A by removing row i and column j. Cofactors of row 1: $C_{11} = 0$, $C_{12} = 0$, $C_{13} = 0$. Using the cofactor formula, this already guarantees $det(A) = 0$. Rank of A is 2, because there are exactly two independent rows.
Using the big formula:

$$det(A) = x \times 0 \times x + x \times 0 \times 0 + x \times x \times 0 - x \times 0 \times 0 - x \times x \times 0 - x \times 0 \times x = 0$$

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Problem 2. Section 5.2, Problem 17, page 265. Clarification. The matrix A_n is the $n \times n$ matrix as in Example 6 on page 261.

Expanding on the last row:

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = (-1)(-1)^{4+3} \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{vmatrix} + (2)(-1)^{4+4} \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{vmatrix} + 2|B_3| = 2|B_3| + (-1)(-1)^{3+3} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2|B_3| - |B_2|$$

Problem 3. Section 5.2, Problem 23 page 266.

a)Expanding on the last row:

$$\begin{vmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & d_{11} & d_{12} \\ 0 & 0 & d_{21} & d_{22} \end{vmatrix} = -d_{21} \begin{vmatrix} a_{11} & a_{12} & b_{12} \\ a_{21} & a_{22} & b_{22} \\ 0 & 0 & d_{12} \end{vmatrix} + d_{22} \begin{vmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \\ 0 & 0 & d_{11} \end{vmatrix}$$

$$= -d_{21} d_{12} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + d_{22} d_{11} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |A|(d_{11}d_{22} - d_{21}d_{12}) = |A||D|$$

b)There are many counterexamples, for instance matrix P from problem 4 works (Prob 31, page 268).

$$P = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

In fact,$|P| = -1$, but the block determinant would give: $|P| = 0 \times 0 - 0 \times 0 = 0$.

c)There are many counterexamples, for instance matrix the same matrix P above works. Here:

$$|AD - CB| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 \neq -1 = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |P|$$

Problem 4. Section 5.2, Problem 31 page 268.
Clarification. In this problem "exchange" means switching of any two numbers in a permutation. The problem asks to find the minimal number of such exchanges needed to obtain the permutation 4, 1, 2, 3 from 1, 2, 3, 4.

Expanding by cofactors of row 1, we are left with only one term:

$$|P| = (1)(-1)^{1+4}|I_3| = -1$$

where I_3 is the 3×3 identity matrix.
Expanding by the "big formula", notice that only one term of the 4! would survive, namely the one corresponding to the permutation $(4, 1, 2, 3)$. Exactly 3 exchanges are needed to reorder into $(1, 2, 3, 4)$, therefore the permutation is odd and negative. So, $|P| = (-1)(1 \times 1 \times 1 \times 1) = -1$.

For P^2, expanding the "big formula", only one term will survive, namely the corresponding to the permutation (3,4,1,2), which is positive (need 4 exchanges). Therefore, $|P^2| = +1 \neq 0$.

Problem 5. Section 5.3, Problem 5 page 279.
Clarification. In this problem, assume that A is invertible.

By Cramer's Rule, if $det(A) \neq 0$, the solution for $Ax = b$ is given by:

$$x_1 = \frac{det(B_1)}{det(A)}$$

$$x_k = \frac{det(B_k)}{det(A)}$$

for any $k > 1$. Here B_k is A with the k-th column substituted by b. Since b is already the first column of A, $B_1 = A$, and thus $x_1 = 1$. For all other cases $k > 1$, B_k has the first and k-th column equal to b, thus B_k is not invertible and $x_k = 0$ for $k > 1$. Therefore, the only solution is $x = (1, 0, 0)^T$.

Problem 6. Section 5.3, Problem 7 page 279.

According to the cofactor formula, the determninant is a linear combination of all the cofactors of some row of the matrix. If all cofactors are zero, this linear combination is zero. So, $det(A) = 0$ and A has no inverse. If none of the cofactors are zero, then A can still be singular, as the linear combination could give zero. A simple example is given by

$$\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

Problem 7. Section 5.3, Problem 14 page 280.

a) The three zero cofactors are: $C_{21} = C_{31} = C_{32} = 0$, thus C is upper triangular. Since $L^{-1} = \dfrac{C^T}{det(L)}$, L^{-1} is lower triangular.

b) One can easily check that:

$$C_{21} = C_{12} = -(bf - de)$$
$$C_{31} = C_{13} = be - dc$$
$$C_{32} = C_{23} = -(ae - bd)$$

Since C is symmetric, then C^T and S^{-1} are symmetric as well. c) Starting with $Q^{-1} = \dfrac{C^T}{det(Q)}$, and using $Q^T = Q^{-1}$

$$Q^T = Q^{-1} = \frac{C^T}{det(Q)}$$

Taking transpose on both sides

$$Q = \frac{C}{det(Q)}$$
$$C = \pm Q$$

Since $det(Q) = \pm 1$ when Q is orthogonal.

Problem 8. Section 6.1, Problem 4 page 293.

First we compute the eigenvalues for $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ by solving $|A - \lambda I| = 0$:

$$\begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda + 6 = (\lambda + 3)(\lambda - 2) = 0$$

This gives two values for λ, $\lambda_1 = -3$ or $\lambda_2 = 2$. We compute the eigenvectors by finding the nullspaces of the matrices $A - \lambda_1 I$ and $A - \lambda_2 I$:

$$(A + 3I) = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \rightarrow (A + 3I)\mathbf{x} = \mathbf{0} \rightarrow \mathbf{x_1} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$(A - 2I) = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \rightarrow (A - 2I)\mathbf{x} = \mathbf{0} \rightarrow \mathbf{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Using the same procedure, you could compute the eigenvalues and eigenvectors for A^2 and then compare. A shortcut is: $A\mathbf{x_1} = \lambda_1 \mathbf{x_1}$ implies $A^2\mathbf{x_1} = \lambda_1 A\mathbf{x_1} = \lambda_1^2 \mathbf{x_1}$. Thus, λ_1^2 is an eigenvalue for A^2 with the same eigenvector $\mathbf{x_1}$. Analogously, λ_2^2 is an eigenvalue for A^2 with eigenvector $\mathbf{x_2}$. Therefore the eigenvalues of A^2 are 9 and 4.

A^2 has the same eigenvectors as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues λ_1^2 and λ_2^2. In this example, $\lambda_1^2 + \lambda_2^2 = 13$, because the sum of the n eigenvalues of a matrix is always equal to its trace.

Problem 9. Section 6.1, Problem 26 page 296. Using the result from Problem $3a$) for the determinant of block matrices:

$$|A-\lambda I| = \left| \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} - \begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I \end{bmatrix} - \right| = \begin{vmatrix} B - \lambda I & C \\ 0 & D - \lambda I \end{vmatrix} = \begin{vmatrix} -1 - \lambda & 3 \\ 2 & -\lambda \end{vmatrix} = |B-\lambda I||D-\lambda I|$$

This expression vanishes whenever $|B - \lambda I| = 0$ or $|D - \lambda I| = 0$. This happens for $\lambda = 1, 2$ and $\lambda = 5, 7$. Therefore, the eigenvalues of A are given by $1, 2, 5, 7$.

Note: to avoid complicating notation we used I for the identity of size 2×2 and 4×4 indistinctly, and we assume you know which is appropriate for each case.

Problem 10. (Computational Problem) See online.