Problem set 6 solutions

Question 1

To find one vector in the plane, we set $x = 1$ and $z = 0$. This forces $y = -1$ and we get $(1, -1, 0)$. Any vector orthogonal to this vector has $x = y$ so set $x = y = 1$. The plane equation forces $z = -1$ and we get $(1, 1, -1)$. $(1, -1, 0)$ and $(1, 1, -1)$ are othogonal and lie in the plane.

$(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$ are orthonormal and lie in the plane.

Question 2

(a)

Suppose $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are othonormal and that $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3 = \mathbf{0}$. Then

$$c_1 = (c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3) \cdot \mathbf{q}_1 = 0.$$

Similarly, $c_2 = c_3 = 0$, so $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are linearly independent.

(b)

Suppose $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are orthonormal.

Let Q be the matrix with i^{th} column given by $\mathbf{q}_i$. Then $Q^TQ = I_n$. A dependency relation

$$x_1\mathbf{q}_1 + \ldots + x_n\mathbf{q}_n = \mathbf{0}$$

can be written in matrix form as $Q\mathbf{x} = \mathbf{0}$. But then $\mathbf{x} = I_n\mathbf{x} = (Q^TQ)\mathbf{x} = Q^T(Q\mathbf{x}) = \mathbf{0}$. The coefficients in the dependancy relation have to be zero, which means that $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are linearly independent.

Question 3

$\mathbf{p} = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T\mathbf{b} =$

$$\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

$\mathbf{e} = \mathbf{b} - \mathbf{p} = (-2, 0, 2)^T$.

$\mathbf{q}_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\mathbf{q}_2 = (-1/\sqrt{2}, 0, 1/\sqrt{2})$.

Question 4

$\mathbf{q}_1 = \mathbf{a} = (1, 0, 0)^T$.
$\mathbf{q}_2 = (\mathbf{b} - 2\mathbf{a})/3 = (0, 0, 1)^T$.
$\mathbf{q}_3 = (\mathbf{c} - 2\mathbf{b})/5 = (0, 1, 0)^T$.

$$Q = A\begin{pmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & -2/5 \\ 0 & 0 & 1/5 \end{pmatrix} \text{ and } A = Q\begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}.$$

Question 5

(a)

It's true for (1×1)-matrices and false for $(n \times n)$ matrices with $n > 1$. Here's a counterexample. Take A to be the matrix with -1 in the $(1,1)$ entry and zeroes everywhere else. Then $\det(I+A) = 0$. However, $1 + \det(A) = 1$ when $n > 1$.

(b)

True. By the product rule we have $|ABC| = |A(BC)| = |A||BC| = |A||B||C|$.

(c)

True for (1×1) matrices or when A has determinant 0. If A is $(n \times n)$, where $n > 1$, with nonzero determinant then $|4A| = 4^n|A| \neq 4|A|$. For a really concrete counterexample to the truth of the general statement, note that when $n > 1$ we have

$$|4I_n| = 4^n \neq 4 = 4|I_n|.$$

(d)

True for (1×1) matrices and false for $(n \times n)$ matrices with $n > 1$.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has determinant 1. Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $(2n \times 2n)$-matrices we can use the block matrices consisting of n copies of A along the diagonal, and n-copies of B along the diagonal, respectively, to find a counterexample. 3×3 counterexample?

Question 6

If $a = b$, $b = c$, or $c = a$, then

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = 0$$

since two of the rows are the same, and thus the matrix is singular. So the formula $(b-a)(c-a)(c-b)$ is true in each of these cases. We now assume that a, b and c are distinct. Then

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & d \end{pmatrix} = (b-a)d,$$

where $d = (c^2-a^2) - \frac{c-a}{b-a}(b^2-a^2) = (c-a)((c+a)-(b+a)) = (c-a)(c-b)$, completing the proof.

Question 7

(a)

We take $A = 1$ and $B = x - \frac{(1,x)}{(1,1)}1$.

$(1,1) = 2\pi$. $(1,x) = \int_0^{2\pi} x dx = 2\pi^2$ so that $B = x - \pi$.

We take $C = x^2 - \frac{(1,x^2)}{(1,1)}1 - \frac{(x-\pi,x^2)}{(x-\pi,x-\pi)}(x - \pi)$.

$(1, x^2) = (x, x) = \int_0^{2\pi} x^2 dx = 8\pi^3/3$ and $(x, x^2) = \int_0^{2\pi} x^3 dx = 4\pi^4$.

So $(x - \pi, x - \pi) = (x, x) - 2\pi(1, x) + \pi^2(1, 1) = 8\pi^3/3 - 4\pi^3 + 2\pi^3 = 2\pi^3/3$ and $(x - \pi, x^2) = (x, x^2) - \pi(1, x^2) = 4\pi^4 - 8\pi^4/3 = 4\pi^4/3$.

Thus $C = x^2 - \frac{4\pi^2}{3}1 - 2\pi(x - \pi) = x^2 - 2\pi x + 2\pi^2/3$.

$(x^2, x^2) = \int_0^{2\pi} x^4 dx = 32\pi^5/5$.

So $(x^2 - 2\pi x + 2\pi^2/3, x^2 - 2\pi x + 2\pi^2/3) =$
$(x^2, x^2) - 4\pi(x, x^2) + (4\pi^2/3)(1, x^2) + 4\pi^2(x, x) - (8\pi^3/3)(1, x) + (4\pi^4/9)(1, 1) =$
$(32/5 - 16 + 32/9 + 32/3 - 16/3 + 8/9)\pi^5 = 8\pi^5/45$.

(b)

$q_1 = A/\sqrt{(A, A)} = 1/\sqrt{(1, 1)} = 1/\sqrt{2\pi}$.
$q_2 = B/\sqrt{(B, B)} = (x - \pi)/\sqrt{(x - \pi, x - \pi)} = (x - \pi)/\sqrt{2\pi^3/3}$.
$q_3 = C/\sqrt{(C, C)} = (x^2 - 2\pi x + 2\pi^2/3)/\sqrt{8\pi^5/45}$.

Question 8

(a)

A_1 is already reduced. A_2 reduces to $\begin{pmatrix} 1 & 1 \\ 0 & 1 - x_1 \end{pmatrix}$. A_3 reduces to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 - x_1 & 1 - x_1 \\ 0 & 0 & 1 - x_2 \end{pmatrix}$.
So $\det A_1 = 1$, $\det A_2 = 1 - x_1$, $\det A_3 = (1 - x_1)(1 - x_2)$ and the pivots are on the diagonal.

(b)

$\det A_n = (1 - x_1)(1 - x_2) \cdots (1 - x_{n-1})$.

(c)

Let e_i be the $1 \times n$ row vector consisting of i zeroes, followed by $n - i$ ones.

For instance, $e_0 = (1, 1, \ldots, 1)$ and $e_1 = (0, 1, 1, \ldots, 1)$ and $e_{n-1} = (0, \ldots, 0, 0, 1)$.

Then A_n reduces to a matrix with first row given by e_0, and where, for $i > 1$, the i^{th} row is given by $(1 - x_{i-1})e_{i-1}$. The pivots are on the diagonal. We can calculate the determinant by multiplying the diagonal entries and this gives the formula we conjectured in part *b*).

Question 9

(a)

We already did this in question 5 part a). Work with $(n \times n)$-matrices where $n > 1$. Take A to be the matrix with -1 in the $(1,1)$ entry and zeroes everywhere else and B to be the identity matrix. Then $\det(A+B) = 0$. However, $\det(A) + \det(B) = 1$.

(b)

We can construct a whole family of such matrices. Let $a, b \in \mathbb{R}$, then

$$\det\left(a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + b\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) = a^2 + b^2 = \det\left(a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) + \det\left(b\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right).$$

For the purpose of answering the question, one may take $a = b = 1$.

Question 10

See other document.