

18.06 PROBLEM SET #5 - SOLUTIONS

FALL 2014

1. Section 4.1, Problem 29, page 205.

Many possible solutions for the first matrix; it's enough to set v as a column and as a row:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & x & x \\ 3 & x & x \end{bmatrix}$$

Many solutions possible for the second matrix; it's enough to have one column equal to v and complete the three rows making them orthogonal to v .

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 0 & -1 \end{bmatrix}$$

From the 6 possible pairs of fundamental subspaces of a matrix, $v \neq 0$ cannot be in the following pairs because they are orthogonal complements: 1. columns space and left nullspace 2. row space and nullspace.

2. Section 4.2, Problem 5, page 214.

Using $P = \frac{aa^T}{a^T a}$ we compute P_1 for a_1 and P_2 for a_2 separately:

$$P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \quad P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

Also $P_1 P_2 = 0$ is the zero matrix, because a_1 and a_2 are orthogonal. Then, for any vector b , $P_2 b$ is a multiple of a_2 and thus also orthogonal to a_1 . Then, $P_1(P_2 b) = 0$ is the zero vector since P_1 is projecting (orthogonal) onto a_1 .

3. Section 4.2, Problem 14, page 215.

Since b is in the column space of A , then the projection will be b itself. In general, $P \neq I$ because for vectors that are not in the column space, their

projection may not be themselves. For $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ we have $A^T A =$

$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$. Then, $P = A(A^T A)^{-1} A^T$ and $p = Pb$ give: $P = \frac{1}{21} \begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$ and

$p = b$.

4. Section 4.2, Problem 17, page 215.

Expanding and using the fact that $P^2 = P$:

$$(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - 2P + P = I - P$$

When P projects onto the column space of A , $I - P$ projects onto the orthogonal complement, which is the left nullspace of A .

5. Section 4.2, Problem 23, page 216.

When A is invertible, A is full rank and thus the column space is the entire

space. P is projecting onto the column space, which now is the entire space. Thus, the projection of every vector b is itself, i.e. $P=I$. The error is $e=b-p=b-b=0$.

6. Section 4.3, Problem 10, page 227.

The system of equations to solve is $Ax=b$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Using Gaussian elimination on the augmented matrix we notice that A has full rank and we can solve for the unique solution by back-substitution:

$$x = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix} \quad \text{Since } A \text{ is invertible, } p=b \text{ and } e=b-p=0. \text{ This agrees with}$$

the fact that the cubic goes through all four points and there is no error in the fitting.

7. Section 4.3, Problem 12, page 228.

Here $b=(b_1, b_2, \dots, b_m)$, $a=(1, 1, \dots, 1)$,

- (a) The system $a^T a \hat{x} = a^T b$ is equivalent to $m\hat{x} = \sum_1^m b_i$. Thus, $\hat{x} = \frac{1}{m} \sum_1^m b_i$ is the average of the b 's.

- (b) The error is given by $e = b - \hat{x}a = \begin{bmatrix} b_1 - \hat{x} \\ b_2 - \hat{x} \\ \dots \\ b_m - \hat{x} \end{bmatrix}$. The standard deviation is

$$\|e\| = \sqrt{\sum_1^m (b_i - \hat{x})^2} \text{ and the variance } \|e\|^2 = \sum_1^m (b_i - \hat{x})^2.$$

- (c) If $b=(1, 2, 6)$ then by a), $\hat{x}=3$, $p=\hat{x}a=(3, 3, 3)$ and $e=b-p=(-2, -1, 3)$. Here $p^T e = -6 - 3 + 9 = 0$, so p is perpendicular to e . The projection

$$\text{matrix is given by } P = \frac{aa^T}{a^T a}, \text{ so } P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

8. Not all matrices satisfying $P^2=P$ are projection matrices. (Matrices satisfying $P^2=P$ are called idempotent matrices). In order to be a projection

matrix it also has to be symmetric. A simple counter example is $P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,

which satisfies $P^2=P$ but is not symmetric.

9. (Note: to avoid notation conflicts, I changed the parameter from A to C).

We would like to fit the three data points $(0, 25)$, $(1, 56)$, $(2, 130)$ with an exponential function of the form $f(t) = Ce^{t}$, where C is the unknown parameter. For every data point we can write an equation that we would like to see satisfied if the exponential goes through all the points:

$$\begin{aligned} f(0) &= Ce^0 = 25 \\ f(1) &= Ce^1 = 56 \\ f(2) &= Ce^2 = 130 \end{aligned}$$

This can be rewritten as a system of equations in the form $A\hat{x}=b$ (here $\hat{x}=[C]$ is just a scalar):

$$\begin{bmatrix} 1 \\ e \\ e^2 \end{bmatrix} [C] = \begin{bmatrix} 25 \\ 56 \\ 130 \end{bmatrix}$$

This system has clearly no solutions: from the first equation $C = 25$, from the second one $C \approx 20.6$, which is impossible. While we cannot solve the original system exactly (and make the exponential go through all the points) we could solve the least squares approximation system. Multiplying both sides of the equation by A^T leads to the so-called "normal equations" $A^T A \hat{x} = A^T b$. In this case it is a single equation, namely:

$$[1 \ e \ e^2] \begin{bmatrix} 1 \\ e \\ e^2 \end{bmatrix} [C] = [1 \ e \ e^2] \begin{bmatrix} 25 \\ 56 \\ 130 \end{bmatrix}$$

$$(1 + e^2 + e^4)C = 25 + 56e + 130e^2$$

Solving for C gives $C \approx 18.06$. While $Ax - b \neq 0$, solving the normal equations gives us a solution $\hat{x} = C$ that minimizes: $\|A\hat{x} - b\|^2$. In terms of the original data, this corresponds to minimize:

$$\|(C, Ce, Ce^2) - (25, 56, 130)\|^2 = |f(0) - 25|^2 + |f(1) - 56|^2 + |f(2) - 130|^2$$

10. See online file for the computational part solutions.