18.06 Problem Set 4. Solutions

Problem 1. Section 3.5, Problem 16, page 180.

Find a basis for each of these subspaces of $\mathbb{R}^4$.

- All vectors whose components are equal.
- All vectors whose components add to zero.
- All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.
- The column space and the nullspace of I (4 by 4).

Solution. These bases are not unique.

- $(1,1,1,1)$.
- $(1,-1,0,0)$, $(1,0,-1,0)$, and $(1,0,0,-1)$.
- $(1,-1,-1,0)$ and $(1,-1,0,-1)$.
- The columns of I are a basis of its column space: $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$. The nullspace contains the zero vector only. By convention, the empty set is the basis of such a space.

Problem 2. Section 3.5, Problem 26, page 181.

Find a basis (and the dimension) for each of these subspaces of 3 by 3 matrices:

- All diagonal matrices.
- All symmetric matrices ($A^T = A$).
- All skew-symmetric matrices ($A^T = -A$).

Solution. The dimensions are 3, 6, and 3 correspondingly. These bases are not unique.

- $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$, and $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$.
- $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$, and $\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$.
- $\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}$, and $\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}$.

Problem 3. Section 3.6, Problem 5, page 191.

If $\boldsymbol{V}$ is the subspace spanned by $(1,1,1)$ and $(2,1,0)$, find a matrix A that has $\boldsymbol{V}$ as its row space. Find a matrix B that has $\boldsymbol{V}$ as its nullspace.

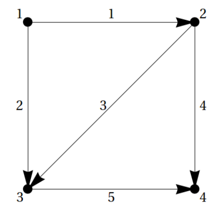
Solution. Matrices A and B are not uniquely defined. We can use the given vectors for rows to find A: $A = \begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}$. Rows of B must be perpendicular to given vectors, so we can use $\begin{bmatrix}1&-2&1\end{bmatrix}$ for B.

Problem 4. Section 3.6, Problem 27, page 194.

If a, b, c are given with $a \neq 0$, how would you choose d so that $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

Solution. To have rank 1, given that the first row is non-zero, the second row should be a multiple of the first row. That is $d = cb/a$. The row space and nullspace should have dimension 1. The first row (a,b) forms the basis of the row space. The nullspace is generated by $(b,-a)$. To show that these two spaces are perpendicular we need to show that the dot product of these two vectors is zero: $(a,b)\cdot(b,-a) = ab - ba = 0$.

Problem 5. Section 8.2, Problem 8, page 429.



Write down the 5 by 4 incidence matrix A for the square graph with two loops. Find one solution to $A\boldsymbol{x} = \mathbf{0}$ and two solutions to $A^T\boldsymbol{y} = \mathbf{0}$.

Solution. The incidence matrix has 4 columns corresponding to 4 vertices and 5 rows corresponding to 5 edges of the graph: $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. The nullspace is generated by all 1s vector: $[1\ 1\ 1\ 1]^T$. The left nullspace is 3-dimensional. The basis can be chosen to match two 3-loops in the graph: $[-1\ 1\ -1\ 0\ 0]^T$ and $[0\ 0\ 1\ -1\ 1]^T$.

Problem 6. Section 8.2, Problem 10, page 429.

Reduce A to its echelon form U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

Solution. The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The nonzero rows of U keep edges 1, 2, 4. Other spanning trees are formed by edges: (1,2,5), (1,3,4), (1,3,5), (1,4,5), (2,3,4), (2,3,5) and (2,4,5).

Problem 7. Section 4.1, Problem 17, page 204.

If $\boldsymbol{S}$ is the subspace of $\mathbb{R}^3$ containing only the zero vector, what is $\boldsymbol{S}^\perp$? If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$, what is $\boldsymbol{S}^\perp$? If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$ and $(1, 1, -1)$, what is a basis for $\boldsymbol{S}^\perp$?

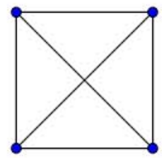
Solution. If $\boldsymbol{S}$ is the subspace of $\mathbb{R}^3$ containing only the zero vector, then $\boldsymbol{S}^\perp$ is $\mathbb{R}^3$. If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$, then $\boldsymbol{S}^\perp$ is the plane spanned by $(1, -1, 0)$ and $(1, 0, -1)$ (this basis is not unique). If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$ and $(1, 1, -1)$, then $\boldsymbol{S}^\perp$ is the line spanned by $(1, -1, 0)$.

Problem 8. Section 4.1, Problem 22, page 204.

If $\boldsymbol{P}$ is the plane of vectors in $\mathbb{R}^3$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\boldsymbol{P}^\perp$. Construct a matrix that has $\boldsymbol{P}$ as its nullspace.

Solution. The vector $(1, 1, 1, 1)$ is a basis for $\boldsymbol{P}^\perp$. The matrix $A = [1\ 1\ 1\ 1]$ has $\boldsymbol{P}$ as its nullspace and $\boldsymbol{P}^\perp$ as row space.

Problem 9. The **complete graph** K_4 is the graph that contains 4 vertices, and any pair of vertices is connected by an edge:

$K_4 =$ 

(a) Show how to pick an ordering of the vertices and edges of this graph and direct arrows on the edges, so that the (signed) incidence matrix becomes

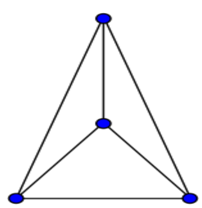
$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(b) Which subgraphs of this graph are spanning trees? (A **spanning tree** is a subgraph without cycles that contains all vertices of the graph.)

Which subsets of rows of A are bases of the row space $C(A^T)$?

(c) Find the dimensions of the 4 fundamental subspaces of A.

(d) We can draw the complete graph K_4 on the plane in a different way so that no pair of edges cross each other:

$K_4 =$ 

(Graphs that can be drawn on the plane without crossing edges are called **planar graphs**.)

Follow the discussion on pages 425–426 of the textbook and explain how the 3 little triangles in this figure give you a basis of the left nullspace $N(A^T)$. Explain how the formula $\dim N(A^T) = m - r$ produces **Euler's formula** for planar graphs.

(e) Find bases of the 4 fundamental subspaces of A.

(f) Calculate the **Laplacian matrix** $L = A^T A$.

(g) Find the dimensions and bases of the 4 fundamental subspaces of the Laplacian matrix $L = A^T A$.

Solution. (a) The vertices of this graph can be numbered in any order. After that the numbering of the edges is uniquely defined.

(b) Any triple of edges except the 4 triples that correspond to loops forms a spanning tree. In the matrix these are rows that do not have an all-zero column. The following 16 triples of edges correspond to spanning trees: $(1, 2, 3)$, $(1, 2, 5)$, $(1, 2, 6)$, $(1, 3, 4)$, $(1, 3, 6)$, $(1, 4, 5)$, $(1, 4, 6)$, $(1, 5, 6)$, $(2, 3, 4)$, $(2, 3, 5)$, $(2, 4, 5)$, $(2, 4, 6)$, $(2, 5, 6)$,

$(3,4,5)$, $(3,4,6)$ and $(3,5,6)$. The same triples correspond to rows that form a basis in the row space of A.

(c) The rank of the matrix, the row space and the column space have dimension 3. Correspondingly the nullspace has dimension 1, and the left nullspace has dimension 3.

(d) Assume that the central node in the planar drawing of the graph has label 1. The three "small loops" (triangles containing the central node) give the following basis of the left nullspace: $(-1,1,0,-1,0,0)$, $(1,0,-1,0,1,0)$, $(0,-1,1,0,0,-1)$. (Usually, "small loops" are called "faces" of a planar graphs.)

The Fundamental Theorem says:

$$\dim N(A) = n - r, \dim N(A^T) = m - r.$$

Since $\dim N(A) = 1$, we get $r = n-1$. Thus $\dim N(A^T) = m - r = m - (n-1)$, that is,

$$n - m + \dim N(A^T) = 1.$$

This is exactly Euler's formula for a planar graph:

$$\#\{vertices\} - \#\{edges\} + \#\{faces\} = 1$$

(e) Any three edges corresponding to a spanning three correspond to rows that form a basis in the row space (see (b) for example). Any three out of four columns form a basis in the column space. The vector $(1,1,1,1)$ forms a basis in the nullspace. The basis for the left nullspace is found in (d): $(-1,1,0,-1,0,0)$, $(1,0,-1,0,1,0)$, $(0,-1,1,0,0,-1)$.

(f) The numbers on the diagonal are the number of edges incident to each vertex. All other entries are -1 symbolizing the fact that all vertices are connected.

$$L = A^T A = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

(g) The rank of L is 3. That means the row space and column space has dimension 3. The nullspaces have dimension 1. Any 3 rows form a basis for row space. Any three columns form a bases for column space. The null space and the left nullspace are spanned by [1 1 1 1].

Problem 10. (Computational Problem)

Available at

`http://web.mit.edu/18.06/www/Fall14/ps4c.pdf`