Solutions to problem set #3

ad problem 1: *Solution to Section 3.2, problem 9.*

(a) False. If the matrix is the zero matrix, then all of the variables are free (there are no pivots).

(b) True. Page 138 says that "if A is invertible, its reduced row echelon form is the identity matrix $R = I$". Thus, every column has a pivot, so there are no free variables.

(c) True. There are n variables altogether (free and pivot variables).

(d) True. The pivot variables are in a 1-to-1 correspondence with the pivots in the RREF of the matrix. But there is at most one pivot per row (every pivot has its own row), so the number of pivots is no larger than the number of rows, which is m.

ad problem 2: *Solution to Section 3.2, problem 12.*

(a) The maximum number of 1's is 11:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) The maximum number of 1's is 13:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ad problem 3: *Solution to Section 3.2, problem 27.*

Assume there exists a 3×3-matrix A whose nullspace equals its column space. In particular, the **dimension** of its nullspace equals the **dimension** of its column space. But the dimension of the nullspace is the number of free variables[1], whereas the dimension of the column space is the number of pivot variables[2]. So A must have as many free variables as it has pivot variables. But the number of free variables and pivot variables altogether is 3, and there is no way to evenly divide a set of 3 variables into two subsets of equal size. So A is impossible.

ad problem 4: *Solution to Section 3.3, problem 2.*

(a) The RREF is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and the rank is 1.

[1] since it is the number of special solutions, but each of these special solutions comes from setting one free variable to 1 and the others to 0

[2] by the second blue box on page 173

(b) The RREF is $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and the rank is 2.

(c) The RREF is $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and the rank is 1.

ad problem 5: *Solution to Section 3.3, problem 10*

For the first matrix, one possible choice is $\mathbf{u} = (3,1,4)^T$ and $\mathbf{v} = (1,2,2)^T$.

For the second matrix, one possible choice is $\mathbf{u} = (2,-1)^T$ and $\mathbf{v} = (1,1,3,2)^T$.

[In principle, one can always choose $\mathbf{v}$ to be the transpose of the first row of A, and $\mathbf{u}$ to be the first column of A divided by the first entry of $\mathbf{v}$. (Unless that first entry is 0; then one needs to choose other rows). We have made some different choices above, however, to avoid fractions.]

ad problem 6: *Solution to Section 3.4, problem 1.*

Step 1: Augmented matrix: $[A \mid b] = \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{pmatrix}$.

Result of Gaussian elimination: $[U \mid c] = \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{pmatrix}$.

The first four columns of this matrix are U; the last column is c.

Step 2: The condition on b_1, b_2, b_3 for $Ax = b$ to have a solution is $b_2 + b_3 - 2b_1 = 0$.

Step 3: **First description:** The column space is the subspace containing all linear combinations of the pivot columns $(2,2,2)^T$ and $(4,5,3)^T$. (The pivots are in columns 1 and 2.)

Second description: The column space contains all vectors b for which $Ax = b$ is solvable, that is, all vectors b for which $b_2 + b_3 - 2b_1 = 0$ (according to Step 2).

Step 4: The pivot variables are x_1 and x_2, and the free variables are x_3 and x_4. The first special solution has $x_3 = 1$ and $x_4 = 0$; by back substitution, one sees that it is $s_1 = (-1,-1,1,0)^T$. The second special solution has $x_3 = 0$ and $x_4 = 1$; by back substitution, one sees that it is $s_2 = (2,-2,0,1)^T$.

Step 5: Time to substitute our values $b_1 = 4$, $b_2 = 3$, $b_3 = 5$. The result of the Gaussian elimination (as obtained in Step 1) then becomes $\begin{pmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The particular solution is obtained by setting both free variables x_3 and x_4 to zero, and then back-substituting; it ends up being $(4,-1,0,0)^T$.

The complete solution thus has the form

$$(4,-1,0,0)^T + c_1 (-1,-1,1,0)^T + c_2 (2,-2,0,1)^T .$$

Step 6: Reduced row echelon form: $[R \mid d] = \begin{pmatrix} 1 & 0 & 1 & -2 & 5/2b_1 - 2b_2 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{pmatrix}$.

Substituting our values of b_1, b_2, b_3 into this, we obtain $[R \mid d] = \begin{pmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

One can recognize the nonzero entries 4 and -1 of the particular solution in the last column d of this, and the negated entries of the special solutions in the R.

ad problem 7: Many answers are possible, and many ways lead to the destination. Below is just one solution.

(a) We are looking for a 3×3-matrix A such that the nullspace of A is the line spanned by the vector $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

This means that $A \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$, and that every solution of $Ax = \mathbf{0}$ is a multiple of $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. Let us forget about the second requirement, and concentrate on $A \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$.

Let us write A as a block matrix: $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, where A_1, A_2 and A_3 are the row-vectors of A. Then, requiring $A \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$ is the same as requiring $A_1 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$, $A_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$ and $A_3 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$. That is, we are looking for three solutions A_1, A_2 and A_3 of the system $(x, y, z) \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$ (the unknowns here are x, y, z). Not every choice of three solutions will do, because there is still the second condition that every solution of $Ax = \mathbf{0}$ is a multiple of $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$; but let us try some choice and hope for the best.

We can rewrite $(x, y, z) \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$ as $3x + z = 0$. This is a system of 1 equation in 3 unknowns x, y, z; we can solve it in any way (Gaussian elimination or RREF as in problem 6), obtaining the two linearly independent solutions $(x, y, z) = (1, 0, -3)$ and $(x, y, z) = (0, 1, 0)$. Let us set $A_1 = (1, 0, -3)$ and $A_2 = (0, 1, 0)$. For A_3, we are uncreative and just set $A_3 = (0, 0, 0)$ (this is a trivial solution but still a solution). Altogether, $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is straightforward to verify that this works – this matrix A indeed has nullspace spanned by the vector $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

(b) We are looking for a 3×3-matrix A such that the nullspace of A is the plane spanned by the vectors $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

This means that $A \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$ and $A \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \mathbf{0}$, and that every solution of $Ax = \mathbf{0}$ is a linear combination of $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Let us forget about the last requirement, and concentrate on $A \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$ and $A \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \mathbf{0}$.

Let us write A as a block matrix: $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, where A_1, A_2 and A_3 are the row-vectors of A. Then, requiring $A \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$ is the same as requiring $A_1 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$, $A_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$ and $A_3 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$. Similarly, we can rewrite $A \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \mathbf{0}$ in terms of A_1, A_2 and A_3. Hence, we are looking for three solutions A_1, A_2 and A_3 of the system

$$\begin{cases} (x,y,z)\begin{pmatrix}3\\0\\1\end{pmatrix}=0; \\ (x,y,z)\begin{pmatrix}1\\1\\2\end{pmatrix}=0 \end{cases}$$

(the unknowns here are x,y,z). Not every choice of three solutions will do, because there is still the condition that every solution of $Ax = \mathbf{0}$ is a linear combination of $\begin{pmatrix}3\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\2\end{pmatrix}$; but let us try some choice and hope for the best.

We can rewrite $(x,y,z)\begin{pmatrix}3\\0\\1\end{pmatrix} = 0$ as $3x+z=0$. Similarly, we can rewrite $(x,y,z)\begin{pmatrix}1\\1\\2\end{pmatrix} = 0$ as $x+y+2z=0$. So we must solve the system

$$\begin{cases} 3x+z=0; \\ x+y+2z=0 \end{cases}.$$

This is a system of 2 equations in 3 unknowns x,y,z; we can solve it in any way (Gaussian elimination or RREF as in problem 6), obtaining the solution $(x,y,z) = (1,5,-3)$. Let us set $A_1 = (1,5,-3)$, $A_2 = (0,0,0)$ and $A_3 = (0,0,0)$. Altogether, $A = \begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix} = \begin{pmatrix}1&5&-3\\0&0&0\\0&0&0\end{pmatrix}$. It is straightforward to verify that this works – this matrix A indeed has nullspace spanned by the vectors $\begin{pmatrix}3\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\2\end{pmatrix}$.

[**Remark for the curious:** Why did the above work? In other words, how did we make sure that in **(a)**, the nullspace of A is really the line spanned by the vector $\begin{pmatrix}3\\0\\1\end{pmatrix}$ in **(a)** (and not a bigger subspace)? We had some freedom in choosing A_1, A_2 and A_3 among the (infinitely many) solutions of $(x,y,z)\begin{pmatrix}3\\0\\1\end{pmatrix} = 0$; it is easy to see that not every choice would have worked (for instance, if we had taken all of A_1, A_2 and A_3 to be the zero vector, then the nullspace of A would be the whole $\mathbf{R}^3$, which is way bigger than the line we want). Did we just have luck with our choice of A_1, A_2 and A_3, or did we follow a generalizable strategy?

It turns out that what we did is a particular case of a generalizable strategy. We solved the system $(x, y, z) \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$, and found that its solution space is spanned by two vectors (namely, $(1, 0, -3)$ and $(0, 1, 0)$). We set A_1 and A_2 to be these vectors, and A_3 to be zero. What we did in part **(b)** was similar – here, the solution space was spanned by one vector only, so we only set A_1 to this vector, and A_2 and A_3 to be zero. The general strategy is to find a basis $(v_1, v_2, ..., v_k)$ of the solution space to the system, then set $A_1 = v_1$, $A_2 = v_2$, ..., $A_k = v_k$, and set all remaining rows of A (that is, A_{k+1}, A_{k+2}, ..., A_n) to zero.

This is not the only strategy that works. Actually it is enough that the rows of A span the whole solution space of the system they are supposed to solve. So "most" choices of A_1, A_2, ..., A_n will work, unless they are "too special" like choosing only zero vectors.]

ad problem 8: (a) We do Gaussian elimination:

$$A = \begin{pmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{pmatrix} \overset{\substack{\text{switch rows} \\ \text{1 and 2}}}{\longrightarrow} \begin{pmatrix} b & c & d \\ 0 & a & 0 \\ 0 & e & 0 \end{pmatrix} \overset{\substack{\text{subtract } e/a \text{ times} \\ \text{row 2 from row 3}}}{\longrightarrow} \begin{pmatrix} b & c & d \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

and then we continue using Gauss-Jordan elimination:

$$\begin{pmatrix} b & c & d \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{\substack{\text{divide row 1} \\ \text{by } b}}{\longrightarrow} \begin{pmatrix} 1 & c/b & d/b \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{\substack{\text{divide row 2} \\ \text{by } a}}{\longrightarrow} \begin{pmatrix} 1 & c/b & d/b \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\substack{\text{subtract } c/b \text{ times} \\ \text{row 2 from row 1}}}{\longrightarrow} \begin{pmatrix} 1 & 0 & d/b \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the end result $\begin{pmatrix} 1 & 0 & d/b \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is the RREF of A.

(b) The rank of A is 2 (being the number of pivots in the RREF).

(c) We are solving $A\mathbf{x} = \mathbf{0}$. This is equivalent to solving $\begin{pmatrix} 1 & 0 & d/b \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}$.

Out of the unknowns x_1, x_2, x_3 making up the solution vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, the first two ($x_1$ and x_2) are pivot variables, while the third (x_3) is a free variable. So there is only one special solution, s_1, obtained by setting $x_3 = 1$ and determining the pivot variables by back substitution. It is $s_1 = \begin{pmatrix} -d/b \\ 0 \\ 1 \end{pmatrix}$.

(d) To solve this we need to augment the matrix A by the column $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and carry this column along in the Gauss-Jordan elimination process:

$$\left[A \mid \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & a & 0 & 0 \\ b & c & d & 1 \\ 0 & e & 0 & 0 \end{pmatrix} \overset{\substack{\text{switch rows} \\ \text{1 and 2}}}{\longrightarrow} \begin{pmatrix} b & c & d & 1 \\ 0 & a & 0 & 0 \\ 0 & e & 0 & 0 \end{pmatrix}$$

$$\overset{\substack{\text{subtract } e/a \text{ times} \\ \text{row 2 from row 3}}}{\longrightarrow} \begin{pmatrix} b & c & d & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\substack{\text{divide row 1} \\ \text{by } b}}{\longrightarrow} \begin{pmatrix} 1 & c/b & d/b & 1/b \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\substack{\text{divide row 2} \\ \text{by } a}}{\longrightarrow} \begin{pmatrix} 1 & c/b & d/b & 1/b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\substack{\text{subtract } c/b \text{ times} \\ \text{row 2 from row 1}}}{\longrightarrow} \begin{pmatrix} 1 & 0 & d/b & 1/b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The particular solution $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of $A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is now obtained by setting the free variable x_3 to 0 and finding the pivot variables by back substitution. It turns out to be $\begin{pmatrix} 1/b \\ 0 \\ 0 \end{pmatrix}$. So the complete solution is

$$\begin{pmatrix} 1/b \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -d/b \\ 0 \\ 1 \end{pmatrix}, \qquad c_1 \in \mathbf{R}.$$

(e) The question is simply for which values of a, b, c, d, e we have $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{0}$. Since $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ b-d \\ 0 \end{pmatrix}$, this is tantamount to asking for which values of a, b, c, d, e we have $\begin{pmatrix} 0 \\ b-d \\ 0 \end{pmatrix} = \mathbf{0}$. The answer is obvious: for the ones that satisfy $b = d$.

ad problem 9: (a) Rank 2, invertible.
(b) Rank 2, non-invertible.
(c) Rank 4, invertible.
(d) Rank 4, non-invertible.
(e) Rank 6, invertible.
(f) Rank 6, non-invertible.
[**For the curious:** There is a pattern here. Let B_n be the $n \times n$ matrix whose entries on the diagonal one step below the main diagonal, and also on the diagonal one step above the main diagonal are 1, while all its other entries are 0. So the matrix in part **(a)** is B_2, the matrix in part **(b)** is B_3, and so on. The general pattern is this: The matrix B_n has rank $\begin{cases} n, & \text{if } n \text{ is even;} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$. One way to see this is by trying to apply Gaussian elimination to B_n (not Gauss-Jordan – this would only overcomplicate things). The first step is to switch the first two rows; this gives us two pivots immediately. Then we subtract the second row from the third. As a result, the matrix looks like this:

$$\begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \cdots \end{pmatrix},$$

where $*$ means entries we don't care about. In block form, this is $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ \mathbf{0} & \mathbf{0} & B_{n-2} \end{pmatrix}$, where $*$ means rows we don't care about. So after these first two steps, we are essentially left with two pivots in the first two columns, and a matrix B_{n-2} (and some entries in the first two rows that we don't need to care about because we are only doing Gauss, not Gauss-Jordan). We then proceed to do Gaussian elimination to the B_{n-2} matrix in the lower right corner. Altogether, the Gaussian elimination does the following to B_n:

- switch row 1 with row 2;
- subtract row 2 from row 3;
- switch row 3 with row 4;
- subtract row 4 from row 5;
- switch row 5 with row 6;
- subtract row 6 from row 7;
- ...

This process ends when we run out of rows. If n is even, it ends with a 1 on the diagonal, so every column has a pivot and the rank is n. If n is odd, then it ends with a 0 on the diagonal, so the last column has no pivot, but all the columns before it have one, so the rank is $n-1$.]

ad problem 10: (a) I shall use Sage:

```
sage: def A(n):
....:     return Matrix(QQ, [[int(abs(i-j) < 2) for i in range(n)]
....:                         for j in range(n)])
....:
sage: A(8).echelon_form()
[ 1  0  0  0  0  0  0  1]
[ 0  1  0  0  0  0  0 -1]
[ 0  0  1  0  0  0  0  0]
[ 0  0  0  1  0  0  0  1]
[ 0  0  0  0  1  0  0 -1]
[ 0  0  0  0  0  1  0  0]
[ 0  0  0  0  0  0  1  1]
[ 0  0  0  0  0  0  0  0]
```

(b) There is only one special solution $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_8 \end{pmatrix}$ (up to scaling); it is obtained by setting $x_8 = 1$ and working one's way up by back substitution. It is $\begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

(c) The pattern is that the nullspace of A_n is:

- zero, if $n+1$ is not divisible by 3;
- one-dimensional, if $n+1$ is divisible by 3, and in this case is spanned by

$$\text{the vector} \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ -1 \\ 1 \end{pmatrix}.$$

[For the curious: This can be proven in a way similar to what we did in problem 9; this time, however, Gaussian elimination does not take our matrix to $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ \mathbf{0} & \mathbf{0} & A_{n-2} \end{pmatrix}$ in two steps, but rather takes it to $\begin{pmatrix} 1 & 1 & 0 & * \\ 0 & 1 & 1 & * \\ 0 & 0 & 1 & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{n-3} \end{pmatrix}$ in three steps, whence the divisiblity-by-3 condition in the result.]