18.06 PROBLEM SET #2 - SOLUTIONS

FALL 2014

1. Section 2.6, Problem 13, page 104.
The four conditions are $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$.
$$L = \begin{bmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 1&1&1&0 \\ 1&1&1&1 \end{bmatrix} \qquad U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

2. Section 2.6, Problem 16, page 105.
Solving $Lc = b$, leads to $c = (4,1,1)^T$ and then solving $Ux = c$ leads to $x = (3,0,1)^T$.
$$A = LU = \begin{bmatrix} 1&1&1 \\ 1&2&2 \\ 1&2&3 \end{bmatrix}$$

3. Section 2.6, Problem 19, page 105.
$$A = \begin{bmatrix} 1&1&0 \\ 1&2&1 \\ 0&1&2 \end{bmatrix} = \begin{bmatrix} 1&0&0 \\ 1&1&0 \\ 0&1&1 \end{bmatrix} \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 0&0&1 \end{bmatrix} = \begin{bmatrix} 1&0&0 \\ 1&1&0 \\ 0&1&1 \end{bmatrix} \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 0&0&1 \end{bmatrix}$$
Assume $a,b,c \neq 0$. Then:
$$A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = \begin{bmatrix} 1&0&0 \\ 1&1&0 \\ 0&1&1 \end{bmatrix} \begin{bmatrix} a&a&0 \\ 0&b&b \\ 0&0&c \end{bmatrix} = \begin{bmatrix} 1&0&0 \\ 1&1&0 \\ 0&1&1 \end{bmatrix} \begin{bmatrix} a&0&0 \\ 0&b&0 \\ 0&0&c \end{bmatrix} \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 0&0&1 \end{bmatrix}$$

4. Section 2.7, Problem 15, page 117.
(a) If P sends row 1 to row 4, then P^T sends row 4 to row 1.
Note: Here, P is not necessarily a row exchange matrix, it is just a permutation matrix. Only when $P^T = P$ the row exchanges come in pairs with no overlap.

(b) Several answers are possible, one of them is: $P = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{bmatrix}$.

5. Section 2.7, Problem 17, page 117.
2x2 symmetric matrices, $A^T = A$, many answers possible:
(a) A is not invertible:
Any matrix of the form $\begin{bmatrix} a&b \\ b&c \end{bmatrix}$ where $b^2 = ac$ works. For instance: $\begin{bmatrix} 1&2 \\ 2&4 \end{bmatrix}$.

(b) A is invertible but cannot be factored into LU (row exchanges needed):
Any matrix of the form $\begin{bmatrix} 0&b \\ b&c \end{bmatrix}$ where $b \neq 0$ works. For instance: $\begin{bmatrix} 0&1 \\ 1&1 \end{bmatrix}$.

(c) A can be factored into LDL^T but not into LL^T (because of negative D):
Any matrix of the form $\begin{bmatrix} a & ab \\ ab & ab^2+c \end{bmatrix}$ where $a \neq 0$ and $c < 0$ works.
For instance: $\begin{bmatrix} 1&2 \\ 2&3 \end{bmatrix} = \begin{bmatrix} 1&0 \\ 2&1 \end{bmatrix} \begin{bmatrix} 1&0 \\ 0&-1 \end{bmatrix} \begin{bmatrix} 1&2 \\ 0&1 \end{bmatrix}$.

6. Section 3.1, Problem 10, page 128.
(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.
Given a vector $v = (x, x, z)$ on that plane and a scalar c, we see that $cv = (cx, cx, cz)$ lies on that plane because $cx = cx$. Given one more vector on the plane $w = (a, a, b)$, we see that $v + w = (x + a, x + a, z + b)$ remains on that plane because the first two components match. Thus YES, this plane is a subspace of $\mathbf{R^3}$.

(b) The plane of vectors (b_1, b_2, b_3) with $b_1 = 1$.
The zero vector $(0, 0, 0)$ needs to be on the plane, for it to be a subspace of $\mathbf{R^3}$. Since this plane does not pass through the origin, it is NOT a subspace of $\mathbf{R^3}$.

(c) The plane of vectors (b_1, b_2, b_3) with $b_1 b_2 b_3 = 0$.
Both $(0, 1, 1)$ and $(1, 0, 0)$ lie on this plane, but not their sum $(1, 1, 1)$. Therefore, this plane is NOT a subspace of $\mathbf{R^3}$.

(d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.
YES, this is definitely a subspace of $\mathbf{R^3}$. In fact, this subspace is the plane that contains both vectors and passes through the origin.

(e) All vectors (b_1, b_2, b_3) that satisfy $b_1 + b_2 + b_3 = 0$.
Actually, $b_1 + b_2 + b_3 = 0$ is the equation of a plane that contains the origin. Therefore, YES, the set of all those vectors (forming a plane) is a subspace of $\mathbf{R^3}$.

(f) All vectors (b_1, b_2, b_3) with $b_1 \leq b_2 \leq b_3$.
Pick a vector in that set, $v = (b_1, b_2, b_3)$ and consider $-v = (-b_1, -b_2, -b_3)$ (multiplying by the scalar $c = -1$). Here, $-b_1 \geq -b_2 \geq -b_3$, so $-v$ does not belong to the set. So, this set is NOT a subspace of $\mathbf{R^3}$.

7. Recall that *the column space consists of all linear combinations of the columns*. If the column space is a just line, then all columns need to be a multiple of, say, the first column. We can also choose that first column to be the given vector $v = (1, 2, 3)^T$. Let us write one symmetric matrix that satisfies the problem:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

8. Pascal's triangle 6×6: $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$

For better visualization, we expand the matrix-vector product as a linear combination of the columns.

$$A(1,1,1,1,1,1)^T = \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&2&1&0&0&0\\ 1&3&3&1&0&0\\ 1&4&6&4&1&0\\ 1&5&10&10&5&1 \end{bmatrix} \begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix} =$$

$$= 1\times\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} + 1\times\begin{bmatrix}0\\1\\2\\3\\4\\5\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\1\\3\\6\\10\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\0\\1\\4\\10\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\0\\0\\1\\5\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\4\\8\\16\\32\end{bmatrix}$$

Note that the components of the last vector are powers of 2.

$$A^2(1,1,1,1,1,1)^T = A(1,2,4,8,16,32)^T = \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&2&1&0&0&0\\ 1&3&3&1&0&0\\ 1&4&6&4&1&0\\ 1&5&10&10&5&1 \end{bmatrix} \begin{bmatrix} 1\\2\\4\\8\\16\\32 \end{bmatrix} =$$

$$= 1\times\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} + 2\times\begin{bmatrix}0\\1\\2\\3\\4\\5\end{bmatrix} + 4\times\begin{bmatrix}0\\0\\1\\3\\6\\10\end{bmatrix} + 8\times\begin{bmatrix}0\\0\\0\\1\\4\\10\end{bmatrix} + 16\times\begin{bmatrix}0\\0\\0\\0\\1\\5\end{bmatrix} + 32\times\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\3\\9\\27\\81\\243\end{bmatrix}$$

Note that the components of the resulting vector for $m=2$ are powers of 3. The same pattern occurs for $m=3$.

$$A^3(1,1,1,1,1,1)^T = \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&2&1&0&0&0\\ 1&3&3&1&0&0\\ 1&4&6&4&1&0\\ 1&5&10&10&5&1 \end{bmatrix} \begin{bmatrix} 1\\3\\9\\27\\81\\243 \end{bmatrix} = \begin{bmatrix} 1\\4\\16\\64\\256\\1024 \end{bmatrix}$$

Based on these observations, we can guess what the result would be for any positive integer $m-1$, namely:

$$A^{m-1}(1,1,1,1,1,1)^T = \begin{bmatrix} 1\\m\\m^2\\m^3\\m^4\\m^5 \end{bmatrix}$$

If this is true for a given $m-1$, the next step, m, $A^m(1,1,1,1,1,1)^T$ is obtained by the product $A(1,m,m^2,m^3,m^4,m^5)^T$. The $n+1$-th row of A is $\left[\binom{n}{0}\ \binom{n}{1}\ \cdots\ \binom{n}{n}\ 0\cdots\ 0\right]$ The $n+1$-th component of the desired vector is the dot product between the corresponding $n+1$-th row of A and $(1,m,m^2,m^3,m^4,m^5)^T$, which gives:

$$\sum_{k=0}^{n} \binom{n}{k} m^k = (m+1)^n$$

by Newton's Binomial Formula.

9. Several possible answers, just showing one for each case as reference.

Order 1: $P = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}$

Only possible answer, $P = I$:

Order 2: $P = \begin{bmatrix} 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \end{bmatrix}$

Note that $P \neq I$ but $P^2 = I$.

Order 3: $P = \begin{bmatrix} 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \end{bmatrix}$

We can check that this permutation matrix is such that $P \neq I$, $P^2 \neq I$ but $P^3 = I$.

Order 4: $P = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}$

We can check that this permutation matrix is such that $P \neq I$, $P^2 \neq I$, $P^3 \neq I$ but $P^4 = I$.

Order 5: $P = \begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \end{bmatrix}$

We can check that this permutation matrix is such that $P \neq I$, $P^2 \neq I$, $P^3 \neq I$, $P^4 \neq I$ but $P^5 = I$.

Order 6: $P = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \end{bmatrix}$

We can check that this permutation matrix is such that $P \neq I$, $P^2 \neq I$, $P^3 \neq I$, $P^4 \neq I$, $P^5 \neq I$ but $P^6 = I$.

You can see that the larger the order, the naive way of checking every power is not efficient. Let us illustrate how you can speed up the determination of the order by using *disjoint cycles*. The last permutation matrix above can be represented as the permutation $\{1,2,3,4,5\} \rightarrow \{2,3,1,5,4\}$. If we start by 1, one step of the permutation takes us to 2, from 2 one more step takes us to 3, and finally back to 1. The cycle (1,2,3) is disjoint of the cycle (4,5). The cycle (1,2,3) is of order 3, because we need 3 permutations to return to the identity, and the cycle (4,5) is of order 2. Since these cycles are disjoint, the order of the original permutation is the least common multiple of the orders of the individual cycles. Thus, the order of the last permutation matrix is 6.

b)10×10 permutation matrix, order 30: we can construct this case by choosing as many disjoint cycles as we need such that the least common multiple of their orders is 30, say three disjoint cycles of orders 2,3, and 5. For example, let us choose the cycles (4,5), (3,10,8), and (1,2,9,6,7) forming the permutation $\{1,2,3,4,5,6,7,8,9,10\} \rightarrow \{2,9,10,5,4,7,1,3,6,8\}$. The corresponding permutation matrix is:

$$P = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

10. See separate file for the computational part solutions.