Solutions to problem set #1

ad problem 1: *Solution to Section 1.2, problem 13.*
Several answers are possible. For example, $(1,0,-1)$ and $(0,1,0)$ fit the bill.
Remarks: A more systematic approach is given by the "Gram-Schmidt process" (§4.4). But in this specific problem, it is enough to just "look".

ad problem 2: *Solution to Section 2.1, problem 7.*
In Problem 5 the columns are $(1,1,2)$ and $(1,2,3)$ and $(1,1,2)$. This is a "singular case" because the third column is a linear combination of the first two.
Find two combinations of the columns that give $\mathbf{b} = (2,3,5)$: There are infinitely many possible answers to this; the simplest is probably "first column + second column" and "second column + third column". These do count as two different combinations, because they have different coefficients in front of the columns.
This is only possible for $\mathbf{b} = (4,6,c)$ if $c = \underline{10}$. (Indeed, this is possible if and only if we can find real numbers x, y, z such that $x(1,1,2) + y(1,2,3) + z(1,1,2) = (4,6,c)$. This is the following system of linear equations in x, y, z:

$$\begin{aligned} x + y + z &= 4; \\ x + 2y + z &= 6; \\ 2x + 3y + 2z &= c. \end{aligned}$$

Gaussian elimination transforms this into

$$\begin{aligned} x + y + z &= 4; \\ y \quad\quad &= 2; \\ y \quad\quad &= c - 8. \end{aligned}$$

This is solvable if and only if $c - 8 = 2$, that is, $c = 10$.)

ad problem 3: *Solution to Section 2.2, problem 12.*
We put pivots in boxes rather than circles. The first pivot is the 2 in the first row:

$$\begin{aligned} \boxed{2}x + 3y + z &= 8; \\ 4x + 7y + 5z &= 20; \\ -2y + 2z &= 0. \end{aligned}$$

We use this pivot to get rid of the 4 in the second row (by subtracting twice the first row from the second row):

$$\begin{aligned} \boxed{2}x + 3y + z &= 8; \\ y + 3z &= 4; \\ -2y + 2z &= 0. \end{aligned}$$

There is now a pivot on the second row:

$$\begin{aligned} \boxed{2}x + 3y + z &= 8; \\ \boxed{1}y + 3z &= 4; \\ -2y + 2z &= 0. \end{aligned}$$

We use it to clear out the -2 on the third row:

$$\begin{aligned} \boxed{2}x + 3y + z &= 8; \\ \boxed{1}y + 3z &= 4; \\ 8z &= 8. \end{aligned}$$

The 8 on the third row is now a pivot:

$$\begin{aligned} \boxed{2}x + 3y + z &= 8; \\ \boxed{1}y + 3z &= 4; \\ \boxed{8}z &= 8. \end{aligned}$$

Back-substitution now gives $z = 1$, $y = 4 - 3 \cdot 1 = 1$ and $x = \frac{1}{2}(8 - 1 - 3 \cdot 1) = 2$. Thus, $(x, y, z) = (2, 1, 1)$ is the unique solution.

ad problem 4: *Solution to Section 2.3, problem 19.*

$$PQ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad QP = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get another non-diagonal matrix M whose square is $M^2 = I$, we can just take $M = -P$. Indeed, $(-P)^2 = (-P)(-P) = PP = P^2 = I$. Other possible answers are $M = PQP = QPQ$ or $M = -Q$ or $M = -PQP$.

ad problem 5: *Solution to Section 2.4, problem 6.*

We have $(A+B)^2 = \begin{pmatrix} 10 & 4 \\ 6 & 6 \end{pmatrix}$ and $A^2 + 2AB + B^2 = \begin{pmatrix} 16 & 2 \\ 3 & 0 \end{pmatrix}$, which are two different matrices.

The correct rule is $(A+B)^2 = A^2 + AB + BA + B^2$. This follows from the distributivity of matrix multiplication.

ad problem 6: *Solution to Section 2.5, problem 11.*

There is a lot of freedom here; the following answers are probably the simplest:
(a) Let $A = (1)$ and $B = (-1)$. Yes, these are 1×1-matrices. But the same idea works for 2×2-matrices ($A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = -A$).

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

ad problem 7: (a) We have

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix};$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

(b) The general rule is that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

for every positive integer n. [1] In particular for $n = 5$, we obtain $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$.

ad problem 8: (a) No. We have $\begin{pmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{pmatrix} \begin{pmatrix} d \\ 0 \\ -b \end{pmatrix} = \mathbf{0}$, so if the matrix was invertible, then the vector $\begin{pmatrix} d \\ 0 \\ -b \end{pmatrix}$ would be $\mathbf{0}$, so that both b and d would be 0, but then we would have $\begin{pmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \mathbf{0}$, which would contradict invertibility.

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[1] This can be proven by induction over n: If $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$, then

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$
$$= \begin{pmatrix} f_{n+1} + f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} \qquad \text{by the recursion.}$$

This is not required of the students.

(b) Yes. For example, it is invertible for $a = 1$, $b = 0$, $c = 1$, $d = 0$ and $e = 1$. (Its inverse is itself in this case.)

ad problem 9: **(a)** Yes. A failure at the first step happens precisely if the first coefficient of the first equation is 0. For example:

$$\begin{aligned} y + z &= 0; \\ x + y + z &= 1; \\ z &= 2. \end{aligned}$$

(One should check that the failure is temporary, not permanent – but this is easy.)

(b) Yes. Here is a simple example:

$$\begin{aligned} x + y + z &= 1; \\ z &= 2; \\ y + z &= 3. \end{aligned}$$

(The first step does nothing, because the second and third rows already have no x'es.)

(c) No. A temporary failure is repaired by switching the row in which it occurs with a row further below. But at the third step, there is no row further below anymore (because we are working with the third row), and thus any failure at the third step must be permanent.

ad problem 10: Sorry, I am using Sage so far.

(a) Function which returns E_n:

```
sage: def E(n):
....:     return Matrix(QQ,
....:                   [[(1 if j >= i else 0) for j in range(n)]
....:                    for i in range(n)])
....:
```

Testing:

```
sage:  E(2)
[1 1]
[0 1]
sage:  E(3)
[1 1 1]
[0 1 1]
[0 0 1]
```

We can get M_n by noticing that $M_n = E_n^2$:

```
sage:  E(2) ** 2
[1 2]
```

```
[0 1]
sage:  E(3) ** 2
[1 2 3]
[0 1 2]
[0 0 1]
sage:  E(4) ** 2
[1 2 3 4]
[0 1 2 3]
[0 0 1 2]
[0 0 0 1]
```

The problem does not ask why this works, but let us explain it all the same. We need to prove that $M_n = E_n^2$. Let $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n\}$.

- Assume first that $j \geq i$ (so the box (i,j) is on the main diagonal or above it). Then, $\left(\text{the } (i,j)\text{-th entry of } M_n\right) = j - i + 1$. What is the (i,j)-th entry of E_n^2 ? It is

$$\begin{aligned}
&\left(\text{the } (i,j)\text{-th entry of } E_n^2\right) \\
&= \left(\text{the } (i,j)\text{-th entry of } E_n \cdot E_n\right) \\
&= \sum_{k=1}^{n} \left(\text{the } (i,k)\text{-th entry of } E_n\right) \cdot \left(\text{the } (k,j)\text{-th entry of } E_n\right) \\
&\qquad\qquad (\text{by the ``row } \cdot \text{ column'' definition of the product of two matrices}) \\
&= \sum_{k=1}^{i-1} \underbrace{\left(\text{the } (i,k)\text{-th entry of } E_n\right)}_{=0} \cdot \underbrace{\left(\text{the } (k,j)\text{-th entry of } E_n\right)}_{=1} \\
&+ \sum_{k=i}^{j} \underbrace{\left(\text{the } (i,k)\text{-th entry of } E_n\right)}_{=1} \cdot \underbrace{\left(\text{the } (k,j)\text{-th entry of } E_n\right)}_{=1} \\
&+ \sum_{k=j+1}^{n} \underbrace{\left(\text{the } (i,k)\text{-th entry of } E_n\right)}_{=1} \cdot \underbrace{\left(\text{the } (k,j)\text{-th entry of } E_n\right)}_{=0} \\
&= \underbrace{\sum_{k=1}^{i-1} 0 \cdot 1}_{=0} + \underbrace{\sum_{k=i}^{j} 1 \cdot 1}_{=j-i+1} + \underbrace{\sum_{k=j+1}^{n} 1 \cdot 0}_{=0} = j - i + 1,
\end{aligned}$$

  which is exactly what we obtained for the (i,j)-th entry of M_n. So the matrices M_n and E_n^2 have the same (i,j)-th entry.

- A similar argument shows that the same result holds (i.e., the matrices M_n and E_n^2 have the same (i,j)-th entry) when we have $j < i$ instead of $j \geq i$, but now the entries are both 0.

So the matrices M_n and E_n^2 are identical entry by entry, which shows that $M_n = E_n^2$.

(b) We can get S_n using $S_n = E_n^3$:

```
sage:  E(2) ** 3
[1 3]
[0 1]
sage:  E(3) ** 3
[1 3 6]
[0 1 3]
[0 0 1]
sage:  E(4) ** 3
[ 1  3  6 10]
[ 0  1  3  6]
[ 0  0  1  3]
[ 0  0  0  1]
```

We need to explain why this is true. We can proceed just as in **(a)**, showing that $S_n = E_n^3$ by comparing the two matrices entry by entry. Let $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n\}$.

- Assume first that $j \geq i$ (so the box (i, j) is on the main diagonal or above it). Then, (the (i, j)-th entry of S_n) $= \frac{(j - i + 1)(j - i + 2)}{2}$. What is the (i, j)-th entry of E_n^3 ? Since $E_n^3 = E_n^2 E_n = M_n E_n$ (because we already know that $E_n^2 = M_n$), it is

$$\begin{aligned}
&\left(\text{the } (i,j)\text{-th entry of } E_n^3\right) \\
&= \left(\text{the } (i,j)\text{-th entry of } M_n \cdot E_n\right) \\
&= \sum_{k=1}^{n} \left(\text{the } (i,k)\text{-th entry of } M_n\right) \cdot \left(\text{the } (k,j)\text{-th entry of } E_n\right) \\
&\qquad\qquad \text{(by the ``row} \cdot \text{column'' definition of the product of two matrices)} \\
&= \sum_{k=1}^{i-1} \underbrace{\left(\text{the } (i,k)\text{-th entry of } M_n\right)}_{=0} \cdot \underbrace{\left(\text{the } (k,j)\text{-th entry of } E_n\right)}_{=1} \\
&+ \sum_{k=i}^{j} \underbrace{\left(\text{the } (i,k)\text{-th entry of } M_n\right)}_{=k-i+1} \cdot \underbrace{\left(\text{the } (k,j)\text{-th entry of } E_n\right)}_{=1} \\
&+ \sum_{k=j+1}^{n} \underbrace{\left(\text{the } (i,k)\text{-th entry of } M_n\right)}_{=k-i+1} \cdot \underbrace{\left(\text{the } (k,j)\text{-th entry of } E_n\right)}_{=0} \\
&= \underbrace{\sum_{k=1}^{i-1} 0 \cdot 1}_{=0} + \underbrace{\sum_{k=i}^{j} (k-i+1) \cdot 1}_{=\sum_{k=i}^{j}(k-i+1)} + \underbrace{\sum_{k=j+1}^{n} (k-i+1) \cdot 0}_{=0} \\
&= \sum_{k=i}^{j} (k-i+1) = \sum_{k=1}^{j-i+1} k = \frac{(j-i+1)(j-i+2)}{2},
\end{aligned}$$

which is exactly what we obtained for the (i,j)-th entry of S_n. So the matrices S_n and E_n^3 have the same (i,j)-th entry.

- A similar argument shows that the same result holds (i.e., the matrices S_n and E_n^3 have the same (i,j)-th entry) when we have $j < i$ instead of $j \geq i$, but now the entries are both 0.

So the matrices S_n and E_n^3 are identical entry by entry, which shows that $S_n = E_n^3$.

(c) The $n \times n$-matrix formed of the tetrahedral numbers in the same way as S_n was formed of triangular matrix is E_n^4. To prove this, one can proceed in the same way as we proved $S_n = E_n^3$ above, but instead of using that the n-th triangular number is the sum of the first n positive integers, we need to use that the n-th tetrahedral number is the sum of the first n triangular numbers.

(d)

```
sage: A = Matrix(QQ, [[1], [2], [3], [1]])
sage: B = Matrix(QQ, [-1, 2, 1, 4])
```

```
sage: (A*B)**10
[-1000000000  2000000000  1000000000  4000000000]
[-2000000000  4000000000  2000000000  8000000000]
[-3000000000  6000000000  3000000000 12000000000]
[-1000000000  2000000000  1000000000  4000000000]
sage: A*B
[-1  2  1  4]
[-2  4  2  8]
[-3  6  3 12]
[-1  2  1  4]
sage: B*A
[10]
```

The trick here is that

$$(AB)^{10} = \underbrace{AB \cdot AB \cdot AB \cdots AB}_{10 \text{ times}} = A \underbrace{(BA \cdot BA \cdot BA \cdots BA)}_{9 \text{ times}} B = A (BA)^9 B.$$

But $BA = (10)$ (a 1×1-matrix) in our case; that is, $BA = 10I$, so that $(BA)^9 = 10^9 I$. Thus, $(AB)^{10} = A \underbrace{(BA)^9}_{=10^9 I} B = A \cdot 10^9 I \cdot B = 10^9 AIB = 10^9 AB$, which is easy to compute by hand.