18.06 Problem Set 10

due Monday, December 01, 2014, before 4:00 pm (sharp deadline) in Room E17-131

The due date is **extended till Monday** because of Thanksgiving vacation.

Problem 1. Let A be a matrix with SVD $A = U\Sigma V^T$.
(a) Find an SVD of A^T.
(b) Find an SVD of A^{-1} if A is invertible.

Problem 2. Find an SVD for each of the following matrices:

$$\text{(a)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 3. A matrix can have several different SVDs. Let A be an $n \times n$ matrix with n different singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_n$ (all σ_i's are distict and nonzero). Explain why all SDVs $A = U\Sigma V^T$ are obtained from each other by multiplying some columns of U by -1 and simultaneously multiplying the same columns of V by -1.

Problem 4. Find the set of all real numbers u such that the matrix $\begin{pmatrix} 1 & u & 0 \\ u & 1 & u \\ 0 & u & 1 \end{pmatrix}$ is positive-definite.

Problem 5. If two symmetric matrices A and B are similar, then show that there exists an **orthogonal** matrix M such that $B = MAM^{-1}$. (Hint: Diagonalize A and B and compare.)

Problem 6. (a) If at least one of the two $n \times n$ matrices A and B is invertible, then show that AB is similar to BA.

(b) Is it true that AB is similar to BA for any two $n \times n$ matrices? (Hint: Find two matrices such that AB is the zero matrix, but BA is not zero.)

Problem 7. Consider the following operations on the (infinite dimensional) space of polynomials $f(x)$. Which of them are linear transformations?

(a) $T_1(f) = f(x) + 2$.
(b) $T_2(f) = 2\,f(x)$.
(c) $T_3(f) = f(x+2)$.
(d) $T_4(f) = f(2x)$.
(e) $T_5(f) = f(x^2)$.
(f) $T_6(f) = (f(x))^2$.
(g) $T_7(f) = f''(x)$.
(h) $T_8(f) = x^2\,f(x)$.

Problem 8. Let V be the vector space of quadratic polynomials $V = \{a\,x^2 + b\,x + c\}$, and W be the vector space of cubic polynomials $W = \{\alpha\,x^3 + \beta\,x^2 + \gamma\,x + \delta\}$.

Let $T : V \to W$ be the linear transformation given by

$$T : f(x) \to (2+x)f(x) + \int_0^x f(t)\,dt.$$

Find the matrix of T with respect to the basis $1, x, x^2$ of V and the basis $1, x, x^2, x^3$ of W.

Problem 9. Let $T : V \to V$ be the linear transformation, whose input and output space V is the space of 2×2 matrices, given by $T(A) = A^T$. Find the matrix of the linear transformation T in the basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 10. Consider the subspace of vectors (x, y, z) in $\mathbb{R}^3$ such that $x+y+z = 0$. Find a basis for this subspace, and describe the linear transformation

$$T : (x, y, z) \mapsto (y, z, x)$$

on this subspace by a matrix with respect to this basis.