Exam Solutions

Problem 1

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of A.

(b) Solve the initial value problem $d\mathbf{u}/dt = A\mathbf{u}$, $\mathbf{u}(0) = (1, -1)^T$.

(c) Find a diagonal matrix which is similar to the matrix A.

(d) Find the singular values σ_1 and σ_2 of A.

(e) Is the matrix A positive definite?

Solutions:

(a) $\det(xI - A) = x^2 - 1$ has roots $\lambda_1 = 1$ and $\lambda_2 = -1$.

(b) $\mathbf{u}(t) = (e^{-t}, -e^{-t})^T$.

(c) Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2}$. Then Q is orthogonal and $A = Q\Lambda Q^{-1}$, where $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) Let $U = Q\Lambda$ and $V = Q$. U and V are orthogonal, $A = UIV^T$ is a singular value decomposition of A, and we see that the singular values are $\sigma_1 = |\lambda_1| = 1$ and $\sigma_2 = |\lambda_2| = 1$.

(e) No, since $\lambda_2 = -1 < 0$.

Problem 2

True or false? If your answer is "true", explain why. If your answer is "false", give a counterexample.

(a) Every positive definite matrix is nonsingular.

(b) If A is an $n \times n$ matrix with real eigenvalues and with n linearly independent eigenvectors which are orthogonal to each other, then A is symmetric.

(c) If a matrix B is similar to A, then B has the same eigenvectors as A.

(d) Any symmetric matrix is similar to a diagonal matrix.

(e) Any matrix which is similar to a diagonal matrix is symmetric.

Solutions:

(a) True: if A is singular, then there exists a nonzero x with $Ax = 0$; thus $x^T Ax = 0$, and A is not positive definite.

(b) True: if the given eigenvectors are $v_1, \ldots, v_n$, and their eigenvalues are $\lambda_1, \ldots, \lambda_n$, respectively, then $A = Q\Lambda Q^T$, where

$$Q = \left(v_1/|v_1| \middle| \cdots \middle| v_n/|v_n| \right) \text{ and } \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n);$$

thus, $A^T = (Q\Lambda Q^T)^T = (Q^T)^T \Lambda^T Q^T = Q\Lambda Q^T = A$.

(c) False: we saw in 1) that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvectors $(1,1)^T$ and $(1,-1)^T$; we also saw that it is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has eigenvectors $(1,0)^T$ and $(0,1)^T$.

(d) True: this is the $Q\Lambda Q^T = Q\Lambda Q^{-1}$ decomposition.

(e) False: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ has distinct eigenvalues 0 and 1, and so it is similar to a diagonal matrix, but it is not symmetric.

Problem 3

Let $A = \begin{pmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{pmatrix}$

(a) Find all values of t, for which the matrix A has 3 nonzero eigenvalues.

(b) Find all values of t, for which the matrix A has 3 positive eigenvalues.

(c) Find all values of t, for which the matrix A has 3 negative eigenvalues.

(d) Find a singular value decomposition $B = U\Sigma V^T$ for the matrix $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(e) Find orthonormal bases of the 4 fundamental subspaces of the matrix B from part (d).

Solutions:

(a) A has 3 nonzero eigenvalues whenever $\det A = 8 - 4t^2$ is nonzero, i.e. whenever $t \neq \pm\sqrt{2}$.

(b) Since $2 > 0$, A has 3 positive eigenvalues whenever $\det A = 8 - 4t^2$ and $\det \begin{pmatrix} 2 & t \\ t & 2 \end{pmatrix} = 4 - t^2$ are positive, i.e. whenever $-\sqrt{2} < t < \sqrt{2}$.

(c) Since $\mathrm{tr} A = 6 > 0$, there are no values of t for which A has 3 negative eigenvalues.

(d) Use `http://web.mit.edu/18.06/www/Fall14/Recitation10_Michael.pdf` algorithm.

  (a) $BB^T = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ has orthonormal eigenvectors $u_1 = (1,1)^T/\sqrt{2}$ and $u_2 = (1,-1)/\sqrt{2}$ with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 0$.

  (b) $\sigma_1 = \sqrt{\lambda_1} = 2$ and $v_1 = B^T u_1/2 = (1,0,1)^T/\sqrt{2}$.

  (c) We can choose $v_2 = (1,0,-1)/\sqrt{2}$ and $v_3 = (0,1,0)$, since they give an orthonormal basis for $N(B^T B) = N(B)$.

  (d) This gives $B = U\Sigma V^T$ where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} /\sqrt{2},\ \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \text{and } V = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{pmatrix} /\sqrt{2}.$$

(e) $C(B) = \left\langle (1,1)^T/\sqrt{2} \right\rangle$, $N(B) = \left\langle (1,0,-1)^T/\sqrt{2},\ (0,1,0)^T \right\rangle$,
$C(B^T) = \left\langle (1,0,1)^T/\sqrt{2} \right\rangle$, $N(B^T) = \left\langle (1,-1)^T/\sqrt{2} \right\rangle$.

Problem 4

Consider the following operations on the space of quadratic polynomials $f(x) = ax^2 + bx + c$. Which of them are linear transformations? If they are linear transformations, find their matrices in the basis $1, x, x^2$. If they are not linear transformations, explain it using the definition of linear transformation.

(a) $T_1(f) = f(x) - f(1)$.

(b) $T_2(f) = f(x) - 1$.

(c) $T_3(f) = x - f(1)$.

(d) $T_4(f) = x^2 f(1/x)$.

(e) The linear transformation $R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the reflection with respect to the line $x + y = 0$. Find the matrix of R in the standard basis of $\mathbb{R}^2$.

Solutions:

(a) T_1 is linear. Its matrix is $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) T_2 is not linear because $T_2(0) = -1 \neq 0$.

(c) T_3 is not linear because $T_3(0) = x \neq 0$.

(d) T_4 is linear. Its matrix is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(e) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.