Exam Solutions

Problem 1

(a) Do Gram-Schmidt orthogonalization for the vectors $a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(b) Find the $A = QR$ decomposition for the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

(c) Find the projection of the vector $(1, 0, 0)^T$ onto the line spanned by the vector $(1, 1, 1)^T$.

(d) Find the projection of the vector $(1, -1, 0)^T$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$.

(e) Find the least squares solution $\hat{x}$ for the system $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \end{pmatrix}$.

Solutions:

(a) a_1 and a_2 are already orthogonal so $b_1 = a_1$ and $b_2 = a_2$.

$$b_3 = a_3 - \frac{a_3 \cdot b_1}{b_1 \cdot b_1} b_1 - \frac{a_3 \cdot b_2}{b_2 \cdot b_2} b_2 = a_3 - 2a_1 - 2a_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Gram-Schmidt orthogonalization on $a_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ gives $b_1 = a_1$ and $b_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ so $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Inspection gives $R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) $$\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} /3.$$

(d) The vector already lies in the plane so projection does nothing: $(1, -1, 0)^T$.

(e) We must solve $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \hat{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \hat{x} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$.

So $\hat{x} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Problem 2

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

(a) Calculate $\det(A)$.

(b) Explain why A is an invertible matrix. Find the $(2,3)$ entry of the inverse matrix A^{-1}.

(c) Notice that all sums of entries in rows of A are the same. Explain why this implies that $(1,1,1)^T$ is an eigenvector of A. What is the corresponding eigenvalue λ_1.

(d) Find two other eigenvalues λ_2 and λ_3 of A.

(e) Find the projection matrix P for the projection onto the column space of A.

Solutions:

(a) Using row operations we see that $\det(A) = \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix}$. Moreover, using the cofactor formula, $\det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} = -3 - 1 = -4$.

(b) $\det(A) = -4 \neq 0$. Matrices with non-zero determinants are invertible. The $(2,3)$ entry of A^{-1} is given by

$$\frac{C_{3,2}}{\det A} = \frac{1}{4} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -1/4.$$

(c) $A(1,1,1)^T = 4(1,1,1)^T$ shows directly that $(1,1,1)^T$ is an eigenvector for A with eigenvalue $\lambda_1 = 4$.

(d) We have $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A) = 4$ and $\lambda_1\lambda_2\lambda_3 = \det(A) = -4$. Remembering that $\lambda_1 = 4$ this gives $\lambda_2 + \lambda_3 = 0$ and $\lambda_2\lambda_3 = -1$. Up to reordering, this system of equations has a unique solution, $\lambda_2 = 1$, $\lambda_3 = -1$.

(e) Since $\det(A) \neq 0$, A is invertible and so the column space of A is all of $\mathbb{R}^3$. The projection matrix onto $\mathbb{R}^3$ is the identity I.

Problem 3

(a) Calculate the area of a triangle on the plane $\mathbb{R}^2$ with the vertices $(1, 0)$, $(0, 1)$, $(3, 3)$ using the determinant.

(b) Find all values of x for which the matrix $A = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$ has an eigenvlue equal to 2.

(c) Diagonalize the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(d) Calculate the power B^{2014} of the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(e) Let Q be any matrix which is symmetric and orthogonal. Find Q^{2014}. Explain your answer.

Solutions:

(a) Translation by $(-1, 0)$ is an isometry and so it is equivalent to find the area of a triangle with the vertices $(0, 0)$, $(-1, 1)$, $(2, 3)$. This is given by

$$\frac{1}{2}\left| \det \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} \right| = \frac{5}{2}.$$

(b) A has an eigenvalue equal to 2 if and only if the matrix $A - 2I$ is singular. Thus, A has an eigenvalue equal to 2 if and only if $\det(A - 2I) = 0$. But

$$\det(A - 2I) = \det \begin{pmatrix} -1 & x \\ 1 & -1 \end{pmatrix} = 1 - x.$$

So $\det(A - 2I) = 0$ if and only if $1 - x = 0$, i.e. $x = 1$.

(c) Since B is diagonal its eigenvalues can be read off from the diagonal $\lambda_1 = 1$ and $\lambda_2 = -1$. We find corresponding eigenvectors $(1, 0)^T$ and $(1, -1)^T$. So $B = S\Lambda S^{-1}$, where

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

By chance we have $S = S^{-1}$.

(d) $B^2 = S\Lambda^2 S^{-1} = SIS^{-1} = I$, so $B^{2014} = (B^2)^{1007} = I$.

(e) Since Q is orthogonal we have $Q^T Q = I$. Since Q is symmetric we have $Q^T = Q$. Thus

$$Q^2 = QQ = Q^T Q = I \text{ and } Q^{2014} = (Q^2)^{1007} = I.$$

Problem 4

Consider the Markov matrix $A = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 1/3 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix}$.

(a) Three of the eigenvalues are $1, 0, -1/3$. Find the fourth eigenvalue of A.

(b) Find the determinant $\det(A)$.

(c) Find the eigenvector of the transposed matrix A^T with eigenvalue $\lambda_1 = 1$.

(d) Find the eigenvector of the matrix A with the eigenvalue $\lambda_1 = 1$. (Hint: notice that the nonzero entries in each column of A are the same.)

(e) Find the limit of $A^k(1,0,0,0)^T$ as $k \longrightarrow +\infty$.

Solutions:

(a) Since $\mathrm{tr}(A) = 0$ the sum of the eigenvalues are 0. Thus, the fourth eigenvalue must be $-2/3$.

(b) The determinant is the product of the eigenvalues, which is 0.

(c) $(1,1,1,1)A = (1,1,1,1)$ and so the eigenvector of A^T with eigenvalue $\lambda_1 = 1$ is $(1,1,1,1)^T$.

(d) The Markov matrix A corresponds to a random walk on the graph with four nodes 1, 2, 3, 4 connected by the edges $(1,2)$, $(1,3)$, $(2,3)$, $(2,4)$, $(3,4)$. The degrees of the nodes are 2, 3, 3, 2. Thus the vector $(2,3,3,2)^T$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

(e) Let $v_1 = (2,3,3,2)^T$ and let v_2, v_3 and v_4 be eigenvectors for $0, -1/3, -2/3$, respectively. Then there exist $c_1, \dots, c_4 \in \mathbb{R}$ with

$$(1,0,0,0)^T = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4.$$

Thus

$$A^k(1,0,0,0)^T = c_1 v_1 + \frac{(-1)^k c_3}{3^k} v_3 + \frac{(-2)^k c_4}{3^k} v_4 \longrightarrow c_1 v_1, \text{ as } k \longrightarrow +\infty$$

To find c_1 we recall that $(1,1,1,1)A = (1,1,1,1)$. By induction we obtain

$$(1,1,1,1)A^k = (1,1,1,1)$$

and so $(1,1,1,1)A^k(1,0,0,0)^T = (1,1,1,1)(1,0,0,0)^T = 1$. Letting $k \longrightarrow +\infty$ we obtain

$$(1,1,1,1)c_1 v_1 = 1$$

so that $c_1 = 1/((1,1,1,1)v_1) = 1/10$. The answer to the question is $(2,3,3,2)^T/10$.