Exam Solutions

Problem 1

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{pmatrix}$.

(a) Find the $A = LU$ factorization of the matrix A.

(b) Solve the system $Ax = (3, 10, 20)^T$.

Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{pmatrix}$ (obtained by replacing the bottom right entry by the parameter k).

(c) For which values of k is the matrix B singular?

(d) Find all values of k for which the system $Bx = (1, 2, 3)^T$ has infinitely many solutions.

(e) Find all values of k for which the system $Bx = (10, 1, 2014)^T$ has exactly one solution.

Answers:

Let's put the matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{pmatrix}$ into LU form.

We get $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & k-3 \end{pmatrix}$ by substracting 2 lots of row 1 from row 2 and 3 lots of row 1 from row 3.

We get $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & k-7 \end{pmatrix}$ by subtracting 2 lots of row 2 from row 3. We see $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

(a) By setting $k = 10$ we obtain $A = LU$ where $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

(b) We spot that $(1, 1, 1)^T$ is a solution. L and U are invertible since their diagonals are nonzero and so A is invertible. Thus, this is the only solution.

Alternatively, we can solve $Lc = (3, 10, 20)^T$ via substitution to give $c = (3, 4, 3)$ and $Ux = c = (3, 4, 3)^T$ via substitution to give $x = (1, 1, 1)^T$.

(c) B is singular if and only if U is singular. U is singular if and only if the last row is zero, i.e. if $k = 7$.

(d) This equation always has a solution since the first column is $(1, 2, 3)^T$. When $k \neq 7$ the matrix is invertible and there is only one solution. When $k = 7$ the vector $(0, 1, -1)^T$ is in the nullspace, so there are infinitely many solutions: in fact, we can see that the solutions are $(1, 0, 0)^T + c(0, 1, -1)$ for $c \in \mathbb{R}$.

(e) When $k \neq 7$ the matrix B is invertible so there exists exactly one solution.

Problem 2

Which are of the following sets of vectors are vector subspaces of $\mathbb{R}^3$? Explain your answer.

(a) All vectors $(x, y, z)^T$ such that $10x + y + 2014z = 0$

(b) All vectors $(x, y, z)^T$ such that $x + y + z \leq 2014$.

(c) All vectors $(x, y, z)^T$ such that $x + y + z = 0$ AND $x + 2y + 3z = 0$.

(d) All vectors $(x, y, z)^T$ such that $x + y + z = 0$ OR $x + 2y + 3z = 0$.

(e) All vectors $(b_1, b_2, b_3)^T$ such that $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} x = (b_1, b_2, b_3)^T$ has a solution.

Answers:

(a) Yes: this is the null space of the 3×1 matrix $\begin{pmatrix} 10 & 1 & 2014 \end{pmatrix}$.

(b) No: $(1, 0, 0)$ is in the set under consideration but $2015(1, 0, 0)$ is not.

(c) Yes: this is the null space of the 3×2 matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

(d) No: $(-1, 0, 1)$ and $(1, 1, -1)$ are in the set under consideration but their sum $(0, 1, 0)$ is not.

(e) Yes: this is the column space of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Problem 3

Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{pmatrix}$

(a) Find the complete solution of $Ax = 0$.

(b) Find the complete solution of $Ax = (1, 2, 0)^T$.

(c) Find the linear condition(s) on b_1, b_2, b_3 that guarantee that the system $Ax = (b_1, b_2, b_3)^T$ has a solution.

(d) Find the rank of A and dimensions of the four subspaces of A.

(e) Find bases of the four fundamental subspaces of A.

Answers:

Let's put the matrix $\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & b_1 \\ 1 & 2 & 2 & 2 & 3 & b_2 \\ -1 & -2 & 0 & 2 & 3 & b_3 \end{pmatrix}$ into RREF.

We get $\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 1 & 2 & 3 & b_1 + b_3 \end{pmatrix}$ followed by $\begin{pmatrix} 1 & 2 & 0 & -2 & -3 & 2b_1 - b_1 \\ 0 & 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 0 & 2b_1 - b_2 + b_3 \end{pmatrix}$.

(a) We read of the special vectors as

$$x_1 = (-2, 1, 0, 0, 0)^T, \; x_2 = (2, 0, -2, 1, 0)^T \text{ and } x_3 = (3, 0, -3, 0, 1)^T.$$

The complete solution to $Ax = 0$ is $x = c_1 x_1 + c_2 x_2 + c_3 x_3$ for $c_1, c_2, c_3 \in \mathbb{R}$.

(b) Let $x_p = (0, 0, 1, 0, 0)^T$. Then $Ax_p = (1, 2, 0)^T$ so that the complete solution to $Ax = (1, 2, 0)^T$ is $x = x_p + c_1 x_1 + c_2 x_2 + c_3 x_3$ for $c_1, c_2, c_3 \in \mathbb{R}$.

(c) We read of the linear combination of b_1, b_2 and b_3 in the zero row above: $2b_1 - b_2 + b_3 = 0$ is the condition that guarantees $Ax = (b_1, b_2, b_3)^T$ has a solution.

(d) We see there are two pivot columns in RREF(A) and so the rank of A is 2. Thus $\dim C(A) = \dim C(A^T) = 2$, $\dim N(A) = 5 - 2 = 3$, and $\dim N(A^T) = 3 - 2 = 1$.

(e) To give a basis for $C(A)$ we read of the the columns corresponding to pivot columns in RREF: $\{(1, 1, -1)^T, (1, 2, 0)^T\}$. We already computed a basis for $N(A)$ in a): $\{x_1, x_2, x_3\}$. To give a basis for $C(A^T)$ we find two independent rows:

$$\{(1, 2, 1, 0, 0)^T, (1, 2, 2, 2, 3)^T\}.$$

To give a basis for $N(A^T)$ we read off the coefficients of the relation in b): $\{(2, -1, 1)^T\}$.

Problem 4

Which of the following statements are true? Explain your answer.

(a) Matrices A and $R = RREF(A)$ always have the same column space $C(A) = C(R)$.

(b) Matrices A and $R = RREF(A)$ always have the same row space $C(A^T) = C(R^T)$.

(c) If A is an $m \times n$ matrix with linearly independent columns, then $m \geq n$.

(d) If A is an $m \times n$ matrix of rank $r = m$, the the left nullspace $N(A^T)$ contains only the zero vector 0.

(e) If two $m \times n$ matrices A and B have the same 4 fundamental spaces

$$C(A) = C(B),\ N(A) = N(B),\ C(A^T) = C(B^T),\ N(A^T) = N(B^T),$$

then $A = B$.

Answers:

(a) No. The matrices $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $R = REEF(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ have different column spaces.

(b) Yes. Row operations do not alter the row space.

(c) Yes. If A has linearly independent columns, then the column space has dimension n. But the column space is a subspace of $\mathbb{R}^m$, which has dimension m. Thus $n \leq m$.

(d) Yes. $r = m$ tells us that the rows are independent, so that $N(A^T) = 0$. Alternatively, we can note that $\dim N(A^T) = m - r = 0$

(e) No. I and $2I$ have the same fundamental spaces.