# 18.06 PSET 9 SOLUTIONS

FALL 13

1. There are 8 families:

(i) $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ evals 0, 0

(ii) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ evals 1, 1

(iii) $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ evals 1, −1

(iv) $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ evals 0, 2

(v) $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ eval 0

(vi) $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ eval 1

(vii) $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ evals $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

(viii) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ evals 1, 0

2.

a) False. For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ are similar (see previous question). Indeed,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

b) True. An invertible matrix has no eigenvalue 0, but a singular matrix has. ~~one.~~ But similar matrices have the same eigenvalues.

c) False: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

You can find this example by noticing that if $A$ is similar to $-A$, then every eigenvalue of $A$ must be minus another eigenvalue of $A$. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $1$ and $-1$, making it a candidate.

2. cont

d) True. The largest eigenvalue of $A+I$ is the largest eigenvalue of $A$, plus one. Thus $A$ and $A+I$ cannot have the same eigenvalues.

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3. No: if $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$,

but

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

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4. $A^T A = A A^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ Let $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$

Eigenvalues: $\frac{3 \pm \sqrt{5}}{2}$ that is, $\varphi^2$, $\psi^2$

Eigenvectors: $\begin{pmatrix} \varphi \\ 1 \end{pmatrix}$ $\begin{pmatrix} \psi \\ 1 \end{pmatrix}$, so normalized eigenvectors:

$$\begin{pmatrix} \frac{\varphi}{\sqrt{\varphi^2+1}} \\ \frac{1}{\sqrt{\varphi^2+1}} \end{pmatrix}, \begin{pmatrix} \frac{\psi}{\sqrt{\psi^2+1}} \\ \frac{1}{\sqrt{\psi^2+1}} \end{pmatrix} \cancel{\begin{pmatrix} \sqrt{\varphi} \\ \frac{1}{\sqrt{\varphi}} \end{pmatrix} \begin{pmatrix} \sqrt{\psi} \\ \frac{1}{\sqrt{\psi}} \end{pmatrix} \text{ since } \varphi^2+1=\varphi,\ \psi^2+1=\psi}$$

Singular values: $\varphi$, $-\psi$ since $\psi$ is negative.

So

$$V^T = \begin{pmatrix} \frac{\varphi}{\sqrt{\varphi^2+1}} & \frac{1}{\sqrt{\varphi^2+1}} \\ \frac{\psi}{\sqrt{\psi^2+1}} & \frac{1}{\sqrt{\psi^2+1}} \end{pmatrix}$$

4. cont

$$\Sigma = \begin{pmatrix} \varphi & 0 \\ 0 & -\psi \end{pmatrix}$$

Now $\quad Av_1 = \varphi u_1$

$\quad\quad Av_2 = -\psi u_2$

so $\quad \varphi u_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\varphi}{\sqrt{\varphi^2+1}} \\ \frac{1}{\sqrt{\varphi^2+1}} \end{pmatrix} = \begin{pmatrix} \frac{\varphi+1}{\sqrt{\varphi^2+1}} \\ \frac{\varphi}{\sqrt{\varphi^2+1}} \end{pmatrix}$

But $\varphi + 1 = \varphi^2$, so

$$u_1 = \begin{pmatrix} \frac{\varphi}{\sqrt{\varphi^2+1}} \\ \frac{1}{\sqrt{\varphi^2+1}} \end{pmatrix} = v_1$$

Similarly,

$$u_2 = \begin{pmatrix} -\frac{\psi}{\sqrt{\psi^2+1}} \\ -\frac{1}{\sqrt{\psi^2+1}} \end{pmatrix} = -v_2. \quad \text{So:}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\varphi}{\sqrt{\varphi^2+1}} & -\frac{\psi}{\sqrt{\psi^2+1}} \\ \frac{1}{\sqrt{\varphi^2+1}} & -\frac{1}{\sqrt{\psi^2+1}} \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & -\psi \end{pmatrix} \begin{pmatrix} \frac{\varphi}{\sqrt{\varphi^2+1}} & \frac{1}{\sqrt{\varphi^2+1}} \\ \frac{\psi}{\sqrt{\psi^2+1}} & \frac{1}{\sqrt{\psi^2+1}} \end{pmatrix}$$

is the SVD.

5.

$$A^TA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Eigenvalues: $\varphi^2$, $\psi^2$ and $0$

+ve square roots of these = singular values of $A$ = $\varphi, -\psi, 0$

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6. Singular values are unaltered by column exchanges, so the singular values of $A$ are equal to those of

$$A' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$A'$ is symmetric, so its singular values are absolute values of its eigenvalues. Eigenvalues are $\varphi, \psi, 0$

$\therefore$ singular values are $\varphi, -\psi, 0$.

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$$K_1 = A_1^T C_1 A_1 = \begin{pmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}$$

$= c_1E_1 + c_2E_2 + c_3E_3$ where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0)$$

$= a_1^T a_1$ $\quad a_1 =$ 1st row of $A$

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} (1\ -1\ 0) = a_2^T a_2$$

$a_2 =$ 2nd row of $A$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} (0\ -1\ 1) = a_3^T a_3$$

$a_3 =$ 3rd row of $A$.