18.06 (Fall '13) PSet 8 Solutions

Exercise 1. Do Problem 7 from 6.3. Suppose P is the projection matrix onto the $45°$ line $y = x$ in $\mathbb{R}^2$. What are its eigenvalues? If $du/dt = -Pu$ (notice minus sign) can you find the limit of $u(t)$ at $t = \infty$ starting from $u(0) = (3, 1)$?

Solution. Any projection matrix has eigenvalues 1 and 0. In this particular case, the projection is a 2 by 2 matrix with 1-dimensional column space (a $y = x$ line). That means any vector on the line has eigenvalue 1 for the matrix P. For example, a vector $(1, 1)$ will do. The nullspace is also one-dimensional. The nullspace is orthogonal to the column space, so we can choose the vector $(1, -1)$ from the nullspace.

The starting vector $(3, 1)$ can be represented as a linear combination of eigenvectors: $(3, 1) = (2, 2) + (1, -1)$. Now, we need to remember that our matrix is $-P$, so the corresponding eigenvalues for vectors $(2, 2)$ and $(1, -1)$ are -1 and 0. The solution is:

$$u(t) = e^{-t}\begin{bmatrix}2\\2\end{bmatrix} + e^{0t}\begin{bmatrix}1\\-1\end{bmatrix}.$$

When t tends to ∞ the result is:

$$\begin{bmatrix}1\\-1\end{bmatrix}.$$

Exercise 2. Do Problem 26 from 6.3. Give two reasons why the matrix exponential e^{At} is never singular:

(a) Write down its inverse.

(b) Write down its eigenvalues. If $Ax = \lambda x$ then $E^{At}x = \underline{\qquad} x$.

Solution.

(a) The inverse of e^{At} is e^{-At}.

(b) If $Ax = \lambda x$ then $e^{At}x = e^{\lambda t}x$.

Exercise 3. Do Problem 31 from 6.3. The *cosine of a matrix* is defined like e^A, by copying the series for $\cos t$:

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots \quad \cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) If $Ax = \lambda x$, multiply each term times x to find the eigenvalue of $\cos A$.

(b) Find the eigenvalues of $A = [\begin{smallmatrix}\pi & \pi \\ \pi & \pi\end{smallmatrix}[$ with the eigenvectors $(1, 1)$ and $(1, -1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $C = \cos A$.

(c) The second derivative of $\cos(At)$ is $-A^2\cos(At)$.

$$u(t) = \cos(At)u(0) \text{ solves } \frac{d^2u}{dt^2} = -A^2u \text{ staarting from } u'(0) = 0.$$

Construct $u(t) = \cos(At)u(0)$ by the usual three steps for that specific A:

1. Expand $u(0) = (4,2) = c_1x_1 + c_2x_2$ in the eigenvectors.
2. Multiply those eigenvectors by _____ and _____ (instead of $e^{\lambda t}$).
3. Add up the solution $u(t) = c_1$_____$x_1 + c_2$_____x_2.

Solution.

(a) If $Ax = \lambda x$, then $(\cos A)x = (\cos\lambda)x$.

(b) The eigenvectors $(1,1)$ and $(1,-1)$ have eigenvalues 2π and 0 correspondingly. Therefore, the eigenvalues of $\cos A$ are $\cos(2\pi) = 1$ and $\cos 0 = 1$. Hence, $C = \cos A = I$.

(c) 1. $u(0) = (4,2) = 3(1,1) + 1(1,-1)$.
2. Multiply those eigenvectors by $\cos(2\pi t)$ and $\cos(0t)$ (instead of $e^{\lambda t}$).
3. Add up the solution $u(t) = 3\cos(2\pi t)(1,1) + (1,-1)$.

Exercise 4. Do Problem 8 from 6.4. If $A^3 = 0$, then the eigenvalues of A must be _____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.

Solution. If λ is an eigenvalue of A, then $\lambda^3 = 0$, therefore, $\lambda = 0$. The simplest example of A, such that A is non-zero and $A^3 = 0$, is $A = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$. If A is diagonalizable, then $A = Q\Lambda Q^T$, where Λ is a diagonalization of A. Then $0 = A^3 = Q\Lambda^3 Q^T$. Hence $\Lambda^3 = 0$. Since Λ is diagonal it follows that $\Lambda = 0$. Therefore, $A = 0$.

Exercise 5. Do Problem 21 from 6.4. True (with reason) or false (with example). "Orthonormal" is not assumed.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(c) The inverse of a symmetric matrix is symmetric.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

Solution.

(a) False. For example, $A = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$.

(b) False. For example, the matrix $A = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$ has one eigenvector, but is not symmetric. On the other hand, if the number of eigenvectors is the same as the size of the matrix, then this is true. Indeed, in this case the eigenvector matrix S is an orthogonal matrix Q, and $A = S\Lambda S^{-1} = Q\Lambda Q^T$, where Λ is diagonal. Then $A^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$.

(c) True. Let us transpose the equality $I = AA^{-1}$, where A is symmetric. We get: $I = I^T = (AA^{-1})^T = (A^{-1})^T A^T = (A^{-1})^T A$. Hence $(A^{-1})^T$ is the inverse of A. Therefore, $(A^{-1})^T = A^{-1}$.

(d) False. For example, a symmetric matrix $A = \left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]$ can have a non-symmetric matrix $\left[\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right]$ as S.

Exercise 6. Do Problem 30 from 6.4. If $\lambda_{\max}$ is the largest eigenvalue of a symmetric matrix A, no diagonal entry can be larger than $\lambda_{\max}$. What is the first entry a_{11} of $A = Q\Lambda Q^T$? Show why $a_{11} \leq \lambda_{\max}$.

Hint: write a_{11} in terms of the entries of Λ and Q, then use the fact that $\lambda_i \leq \lambda_{max}$ for all i.

Solution. $a_{11} = [q_{11} \dots q_{1n}][\lambda_1 \overline{q}_{11} \dots \lambda_n \overline{q}_{1n}]^T \leq \lambda_{\max}(|q_{11}|^2 + \dots + |q_{1n}|^2) = \lambda_{\max})$.

Exercise 7. Do Problem 10 from 6.5. Which 3 by 3 symmetric matrices A and B produce these quadratics?

$$x^T Ax = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3). \text{ Why is } A \text{ positive definite?}$$

$$x^T Bx = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3). \text{ Why is } B \text{ semidefinite?}$$

Solution.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

There are many ways to check this. One way is to check pivots. Pivots of A are 2, 3/2, and 4/3: all positive, so A is positive definite. Pivots of B are 2, 3/2, and 0: all non-negative, so B is semidefinite. Another way is to check upper left determinants. For A they are 2, 5, and 4: all positive. For B they are 2, 5, and 0: all non-negative.

Exercise 8. Do Problem 12 from 6.5. For what numbers c and d are A and B positive definite? Test the 3 determinants:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Solution. Left upper determinants of A are c, $c^2 - 1 = (c-1)(c+1)$, and $c^3 - 3x + 2 = (c-1)^2(c+2)$. They are all positive for $c > 1$. Left upper determinants for B are 1, $d - 4$, and $-4d + 12$. The are never positive at the same time. So B is never positive definite.

Exercise 9. Do Problem 21 from 6.5. For which s and t do A and B have all $\lambda > 0$ (therefore positive definite)? (use the eigenvalues)

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

Solution. The characteristic polynomials of A and B are: $(s - \lambda)^3 - 48(s - \lambda) - 128$ and $(t - \lambda)^3 - 25(t - \lambda)$ correspondingly. The eigenvalues of A are: $s + 4$, $s + 4$ and $s - 8$. They are all positive when $s > 8$. The eigenvalues of B are: t, $t + 5$ and $t - 5$. They are all positive when $t > 5$.

Exercise 10. **Do Problem 3 from 6.5.** For which numbers b and c are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \quad A = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^T.

Solution. The LDL^T forms are:

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ b/c & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & (c^2 - b^2)/c \end{bmatrix} \begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}.$$

The first matrix is positive definite for $-3 < b < 3$, the second for $c > 8$, and the third for both $c > 0$ and $c^2 - b^2 > 0$. The latter can be combined into $c > |b|$.