# PSET 7 18.06 F13

# SOLUTIONS

1. Want $S-P=R-Q$:

$R-Q=(1,1,0)$, so take $S=(2,1,-1)$.

For OPQRSTUV to be a parallelepiped (tilted box) we can take

$$P-O=Q-T=R-U=S-V$$

giving

$$T=Q-P=(0,1,2)$$

$$U=R-P=(1,2,2)$$

$$V=S-P=(1,1,0)$$

This is not the only answer - there are other choices one can make! For instance, we could instead take

$$Q-O=P-T=R-U=S-V$$

etc.

2.

$$P\begin{pmatrix}-2\\1\\0\end{pmatrix} = 0\cdot\begin{pmatrix}-2\\1\\0\end{pmatrix} \quad P\begin{pmatrix}0\\0\\1\end{pmatrix} = 1\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and $$P\begin{pmatrix}1\\2\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}$$

Any linear combination of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\0\end{pmatrix}$ is an eigenvector of eigenvalue 1, so for an eigenvector with no zero components we can take

$$\begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}.$$

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3. a) Yes: rank $B = 2$ (two distinct nonzero eigenvalues)

b) Yes: $\det B^TB = 0$ (det $B=0$ since $B$ has an eigenvalue of 0)

c) No. For example

$B_1 = \begin{pmatrix}1&0&0\\0&2&0\\0&0&0\end{pmatrix}$ $\quad B_1^TB_1 = \begin{pmatrix}1&0&0\\0&4&0\\0&0&0\end{pmatrix}$ has eigenvalues 0, 1 and 4.

$B_2 = \begin{pmatrix}1&1&0\\0&2&0\\0&0&0\end{pmatrix}$ $\quad B_2^TB_2 = \begin{pmatrix}1&1&0\\1&5&0\\0&0&0\end{pmatrix}$ has eigenvalues 0, $3-\sqrt{5}$ and $3+\sqrt{5}$.

also has eigenvalues 0, 1 and 2

3 continued

d) Yes. $B^2$ has eigenvalues $0, 1, 4$ (same eigenvectors)

$B^2+I$ " $1, 2, 5$ "

$(B^2+I)^{-1}$ " $1, \frac{1}{2}, \frac{1}{5}$ "

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4. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ since $a+b=c+d$

To find other eigenvalue, note sum of eigenvalues $=$ trace $= a+d$

$\therefore$ other eigenvalue $= (a+d)-(a+b)$

$= d-b$

$= a-c$

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5. a) $$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

Eqn for eigenvalues of A:

$$\det(A - cI) = -c\left(\tfrac{1}{2}-c\right)-\tfrac{1}{2}$$
$$= c^2 - \tfrac{1}{2}c - \tfrac{1}{2}$$
$$= (c-1)\left(c+\tfrac{1}{2}\right) = 0$$

So A has eigenvalues $1, -\frac{1}{2}$

5. a) continued

$A - I = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue 1

$A + \frac{1}{2} I = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$ so $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector of eigenvalue $-\frac{1}{2}$

b) If $S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

then $A = S \Lambda S^{-1}$

$\therefore \quad A^n = S \Lambda^n S^{-1}$

$$\lim_{n\to\infty} \Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so
$$\lim_{n\to\infty} A^n = S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

c)
$$\lim_{n\to\infty} \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \left(\lim_{n\to\infty} A^n\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

6. ones(n) has diagonal form

$$\begin{pmatrix} n & & 0 \\ & 0 & \\ 0 & & \ddots & \\ & & & 0 \end{pmatrix}$$

so eye(n) + C (ones(n)) has diagonal form

$$\begin{pmatrix} 1+cn & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

We need to find C such that

$$(1+n)(1+cn) = 1.$$

This is satisfied for

$$c = -\frac{1}{1+n}.$$

7. There may be more than one correct answer, but here are some examples:

a) $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ b) $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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8.
- Obviously the square of a matrix with non-negative entries has non-negative entries.
- Let $M$ be a Markov matrix and let $M_{ij}$ be its $i,j$ the entry, so

$$\sum_{i=1}^{n} M_{ij} = 1$$

for any $j$. Then if $N_{ij}$ is the $i,j$ th entry of $M^2$,

$$N_{ij} = \sum_{k=1}^{n} M_{ik} M_{kj}$$

so
$$\sum_{i=1}^{n} N_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} M_{ik} M_{kj}$$
$$= \sum_{k=1}^{n} \Big(\underbrace{\sum_{i=1}^{n} M_{ik}}_{1}\Big) M_{kj}$$
$$= \sum_{k=1}^{n} M_{kj} = 1.$$

9.

Completing $A$ to a Markov matrix gives

$$\begin{pmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{pmatrix}$$

A steady state eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This is always true for a symmetric Markov matrix: if $A$ is any Markov matrix, then $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A^T$ with eigenvalue 1. If $A = A^T$ then $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A$ with eigenvalue 1.

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10. One should observe that the largest eigenvalue of $A^TA$ ~~is~~ scales roughly with $\sqrt{n}$. (In fact, it is asymptotically $2\sqrt{n}$). This, together with any reasonable-looking plots, should suffice for an answer.