18.06 (Fall '13) Problem Set 6 Solutions

1. Problem 3 from 8.5.

   The zero vector is orthogonal and has length 0.

   Alternatively, $(1, -2, 0, 0, 0 \ldots)$ is orthogonal and has length $\sqrt{5}$.

2. Problem 4 from 8.5.

   On $[-1, 1]$, the integrals of any odd function vanishes. So for any c,

$$\int_{-1}^{1} (1)(x^3 - cx)dx = 0 \quad \text{and} \quad \int_{-1}^{1} (1)(x^2 - 1/3)(x^3 - cx)dx = 0.$$

   Choose c so that the remaining integral vanishes:

$$\int_{-1}^{1} (x)(x^3 - cx)dx = \left[x^5/5 - cx^3/3\right]_{-1}^{1} = 2(1/5 - c/3) = 0.$$

   Thus $\boxed{\mathrm{c} = 3/5}$.

3. Problem 7 from 5.1.

   We can use the 2×2 determinant formula: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

   Rotation: $|Q| = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = \boxed{1}$.

   Reflection: $|Q| = (1 - 2\cos^2\theta)(1 - 2\sin^2\theta) - (-2\cos\theta\sin\theta)(-2\cos\theta\sin\theta)$

$$= 1 - 2\cos^2\theta - 2\sin^2\theta + 4\cos^2\theta\sin^2\theta - 4\cos^2\theta\sin^2\theta$$

$$= 1 - 2(\cos^2\theta + \sin^2\theta) = 1 - 2 = \boxed{-1}$$

   (A note on orthogonal matrices) Recall that for any orthogonal matrix we have $QQ^\top = I$. Using the determinant identities $|AB| = |A||B|$ and $|I| = 1$, it follows that $|Q||Q^\top| = |I| = 1$. Finally, we know that $|Q| = |Q^\top|$, so we have $|Q|^2 = 1$ or $|Q| = \pm 1$, a property of orthogonal matrices.

4. Problem 18 from 5.1.

   We first put the matrix into upper triangular form using row operations (without swaps)

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c^2-a^2)-(b^2-a^2)(c-a)/(b-a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{vmatrix}.$$

   Then using the fact that $|A|$ is equal to the product of the diagonals (for A triangular), we arrive at the desired result

$$= \boxed{(b-a)(c-a)(c-b)}\ .$$

5. Problem 29 from 5.1.

   The matrix A need not be square, and therefore $|A|$ is not always computable.

6. Problem 16 from 5.2.

   The 1,1 cofactor of the $n \times n$ matrix is F_{n-1}. The 1,2 cofactor has a 1 in column 1 with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1,2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

7. Problem 12 from 5.2.

   For $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, the cofactor matrix is $\boxed{C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}}$.

   We then find that $\boxed{AC^\top = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$. Since $|A| = 4$, we see that $AC^\top = |A|I$ or $\boxed{A^{-1} = \dfrac{C^\top}{|A|}}$.

8. <u>Problem 28 from 5.3.</u>

The rows are formed by the partials of x, y, z with respect to ρ, ϕ, θ:

$$\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

To get the determinant, expand cofactors along the bottom row:

$$J = \cos\phi(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta) + \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta).$$

Distribute and simplify.

$$= \rho^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$$

$$= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi)$$

$$\boxed{J = \rho^2\sin\phi}.$$

9. We can check orthogonality by performing monte carlo integration in Matlab:

```
x=rand(1e7,1)*2-1;

p0=1+0.*x;
p1=x;
p2=3*x.^2-1;
p3=5*x.^3-3*x;

disp(['p0.*p1 = ' num2str(mean(p0.*p1))]);
disp(['p0.*p2 = ' num2str(mean(p0.*p2))]);
disp(['p0.*p3 = ' num2str(mean(p0.*p3))]);
disp(['p1.*p2 = ' num2str(mean(p1.*p2))]);
disp(['p1.*p3 = ' num2str(mean(p1.*p3))]);
disp(['p2.*p3 = ' num2str(mean(p2.*p3))]);
```

On a single run, the output was

```
p0.*p1 = -0.00017095
p0.*p2 = -0.00025007
p0.*p3 = -0.00027046
p1.*p2 = -0.00029904
p1.*p3 = -0.00020852
p2.*p3 = -0.0002926
```

which are all very close to zero. As we increase the number of samples, the values get even closer to zero, suggesting convergence (and thus orthogonality).

The method above gave us an estimate of the *average* value of the inner product of the two functions over the interval of interest, $[-1, 1]$. To estimate $||p_i||^2 = \int_{-1}^{1} p_i^2(x)\,dx$ we again estimate the average value over the interval, but multiply this result by 2 (the interval width), to get our approximation for the integral.

```
disp(['||p0||^2 = ' num2str(2*mean(p0.*p0))]);
disp(['||p1||^2 = ' num2str(2*mean(p1.*p1))]);
disp(['||p2||^2 = ' num2str(2*mean(p2.*p2))]);
disp(['||p3||^2 = ' num2str(2*mean(p3.*p3))]);
```

which output

```
||p0||^2 = 2
||p1||^2 = 0.6665
||p2||^2 = 1.5993
||p3||^2 = 1.1426
```

and are approximately $||p_0||^2 = 2$, $||p_1||^2 = 2/3$, $||p_2||^2 = 8/5$, and $||p_3||^2 = 8/7$.

We check our results using symbolic integration in Matlab (*not required*):

```
syms z

p0=1;
p1=z;
p2=3*z^2-1;
p3=5*z^3-3*z;

disp(['p0.*p1 = ' char(int(p0*p1,z,-1,1))])
disp(['p0.*p2 = ' char(int(p0*p2,z,-1,1))])
disp(['p0.*p3 = ' char(int(p0*p3,z,-1,1))])
disp(['p1.*p2 = ' char(int(p1*p2,z,-1,1))])
disp(['p1.*p3 = ' char(int(p1*p3,z,-1,1))])
disp(['p2.*p3 = ' char(int(p2*p3,z,-1,1))])

disp(['||p0||^2 = ' char(int(p0*p0,z,-1,1))])
disp(['||p1||^2 = ' char(int(p1*p1,z,-1,1))])
disp(['||p2||^2 = ' char(int(p2*p2,z,-1,1))])
disp(['||p3||^2 = ' char(int(p3*p3,z,-1,1))])
```

which outputs

```
p0.*p1 = 0
p0.*p2 = 0
p0.*p3 = 0
p1.*p2 = 0
p1.*p3 = 0
p2.*p3 = 0

||p0||^2 = 2
||p1||^2 = 2/3
||p2||^2 = 8/5
||p3||^2 = 8/7
```

confirming our result of the monte carlo integration.

10. Here are two possible solutions.

Solution One:

I used the following Matlab function:

```
A = fmincon(@idet,2*rand(5,5)-1,[],[],[],[],-1.*ones(5,5),1.*ones(5,5));
```

This line of code finds a minimum of the function *idet* (defined below) starting from an initial random matrix $A0 = (2 * rand(5,5) - 1)$ and constrains all elements to remain the desired range $[-1, 1]$. The function *idet* is defined as

```
function out = idet(A)
out = 1/abs(det(A));
end
```

which is just the reciprocal of the absolute value of the determinant. So when *idet* is minimized, the absolute value of the determinant is maximized.

Running the code yielded the matrix:

```
A =

     1     1     1    -1     1
    -1     1     1    -1    -1
    -1     1    -1     1     1
     1     1    -1    -1    -1
    -1    -1    -1    -1     1
```

This matrix has a determinant of

```
>> det(A)

ans =

    48
```

This code consistently finds matrices with absolute value of determinant equal to 48, which appears to be the maximum achievable using this method. All such matrices found have all elements equal to either -1 or 1.

Solution Two:

A second approach might be generating a large number of random matrices and storing the best result.

```
A1 = zeros(5,5);
d1 = 0;
for i = 1:1e7
    A = 2*rand(5,5)-1;
    d = abs(det(A));
    if d>d1
        d1 = d;
        A1 = A;
    end
end
```

Running this code found

```
A1 =

    0.7613    0.9799   -0.5793    0.5054    0.9884
    0.5440    0.3285    0.8247    0.9940    0.1550
    0.5361    0.9012   -0.7819   -0.6202   -0.8838
   -0.8076    0.9076   -0.9617    0.4805   -0.8697
   -0.5749    0.9841    0.9334   -0.6644    0.9318

d1 =

   10.0136
```

The second method does considerably worse (and is slower) than the first method.